

Some Developments of the Casimir Effect in p -Cavity of $(D + 1)$ -Dimensional Spacetime

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The Casimir effect for rectangular boxes has been studied for several decades. But there are still some points unclear. Recently, there are new developments related to this topic, including the demonstration of the equivalence of the regularization methods and the clarification of the ambiguity in the regularization of the temperature-dependent free energy. Also, the interesting quantum spring was raised stemming from the topological Casimir effect of the helix boundary conditions. We review these developments together with the general derivation of the Casimir energy of the p -dimensional cavity in $(D + 1)$ -dimensional spacetime, paying special attention to the sign of the Casimir force in a cavity with unequal edges. In addition, we also review the Casimir piston, which is a configuration related to rectangular cavity.

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1. Introduction

The Casimir effect, as the embodiment of the quantum fluctuation, since its first prediction¹ more than 60 years ago, has been put under extensive and detailed study theoretically and experimentally. Yet it still receives increasing attention from the scientific community. The nature of this effect, many aspects of which have been reviewed in a large amount of literature,²⁻⁷ may depend on the background field, the geometry of the configuration, the type of boundary conditions (BCs), the topology of spacetime, the spacetime dimensionality, and the finite temperature. As a simple generalization of the original setup of the two parallel planes,¹ the Casimir effect in rectangular boxes, is one of the frequently considered configuration, and has been a topic for several decades. Various calculation methods have been developed for this configuration. (For an example see Ref. 3 and references therein and also Refs. 7-45).

In the calculation and regularization of Casimir effect inside a rectangular

box, the commonly used methods are Abel-Plana formula and zeta function technique.^{46–56} The zeta function technique,^{46,47,50,51,53,54} which can be traced back to G. H. Hardy,^{57,58} is used to be regarded as an elegant and unique regularization method^{59,60} different from other ones such as frequency cut-off method (see e.g. Ref. 61) and Abel-Plana formula (see e.g. Refs. 62, 63). Although the automatically finite result of this method veils the isolation of the divergent part, the fact that in most cases the outcomes are in agreement with other approaches^{3,4,50,64–73} draws some attention to the investigation of zeta function itself^{3,4,74,75} and its connection with other regularization methods.^{15,76–79} As a matter of fact, the divergent part of zeta function which is implicitly removed is shown and regulated for two and three-dimensional boxes³ by utilization and comparison of the Abel-Plana formula method, which permits explicit separation of the infinite terms. And now, the two methods are proved to be identifiable,⁸⁰ that is, the reflection formula of Epstein zeta function, which is the key of the regularization process, can be derived from the Abel-plana formula. So, one can choose any methods for convenience. We shall come back to the demonstration of the equivalence of these two regularization methods in Sect. 2.

With the help of the powerful and facile technique of zeta function, the dependence of the Casimir effect on the configurations of the box is within reach. Specifically, the attractive or repulsive nature of the force depending on the configuration is a subject of concern in the study of Casimir effect in rectangular boxes,^{3,10–13,16,17,22,24,26–28,52,81–85} and is analysed much conveniently in terms of zeta function. We will give a general derivation of the Casimir energy under various BCs and discuss the result for both equal and unequal edges in Sect. 3, where the previous results in the literature are recovered as special cases. Especially, we review the repulsive force caused by unequal edges under Dirichlet BCs.²²

The calculations in the rectangular box indicate that the Casimir energy may change sign depending not only on the BCs but also on geometry of the configuration. To address the doubt of the repulsive force, the configuration of a rectangular piston, a box divided by an ideal movable partition, is proposed.⁶⁷ For a scalar field obeying Dirichlet BCs on all surfaces, when the separation between the piston and one end of the cavity approaches infinity, the force on the piston is towards another end (the closed end), that is, the force is always attractive, independent of the ratio of the edges.^{39,40} Now it is known that the results are in agreement because the two configurations are actually different.⁸⁶ And then, the Casimir effect on various piston geometries and for various fields under various BCs, and also with various spacetime dimensions attracts a lot of interests.^{31,33,37,39,40,83,84,86–121} Both attractive and repulsive forces are obtained under corresponding conditions. We will focus on the rectangular Casimir piston model of massless and massive scalar field under Dirichlet and hybrid BCs in Sect. 4.

The researches mentioned above are limited to the vacuum state of the quantum field, namely all the excitation are neglected and the temperature of the system is set to be zero, which seems not practical nor feasible. The quantum state contain-

ing particles in thermal equilibrium with a finite characteristic temperature T is a typical situation when considering the influence of temperature on the Casimir effect. Indeed, thermal corrections on the Casimir effect for various configuration did attract a lot of interest.^{12, 25, 35, 36, 38, 48, 49, 122–132} Both controversies and progresses were seen in this topic,^{36, 56, 108, 133–138} and it is exciting that there is a possibility to measure the thermal effect in the Casimir force.^{139–141} The Casimir effect at finite temperature for a p -dimensional rectangular cavity inside a $(D+1)$ -dimensional spacetime was first considered by Ambjørn and Wolfram,¹² and was reconsidered by Lim and Teo³⁶ more recently in detail, expanding the results of different BCs in the low and high temperature regimes. And critical discussion was given by Geyer et al. on the thermal Casimir effect in ideal metal rectangular boxes in three-dimensional space,¹³⁵ pointing out the neglect of the removal of geometrical contributions including the blackbody radiation term in the previous researches, which would lead to the contradiction with the classical limit. Now a common recognition was reached that the terms of order equal to or more than the square of the temperature should be subtracted from the Casimir energy. But the explicit expression of these terms is not easy to get from the calculation of the heat kernel coefficients. Recently, these terms were obtained^{44, 45} by repeatedly using Abel-Plana formula, and more importantly, the subtraction of them was shown clearly by rigorous calculation and regularization of the temperature-dependent part of the thermal scalar Casimir energy and force with different BCs. We will give the review of these results in Sect. 5.

As mentioned before, besides the geometry, the BCs and the temperature, the nontrivial topology of the space can also give rise to the Casimir effect. The scalar field on a flat manifold with topology of a circle S^1 and a Möbius strip may be the simplest examples. Periodic condition $\phi(t, 0) = \phi(t, C)$ caused by the topology of S^1 with circumference of C , and similar antiperiodic condition caused by the topology of the Möbius strip are imposed on the wave function. There are many things in the world that having spring-like structure. For instance, DNA has a double helix structure living in our cells. So, it is interesting to find how could such kind of helix structure would affect the behavior of a quantum field. In fact, it is found that the behavior of the force parallel to the axis of the helix is very much like the force on a spring that obeys the Hooker's law in mechanics when the ratio of the pitch to the circumference of the helix is smaller. However, in this case, the force origins from a quantum effect, and so the helix structure is called a quantum spring, see Ref.¹⁴² for a short review. The Casimir effect for both scalar and fermion fields under helix BCs stem from new types of space topologies was considered.^{143–146} The relation between the two topologies is something like that between a cylindrical and a Möbius strip. The calculation of the Casimir effect under helix BCs in $(D+1)$ -dimensional spacetime shows that there is a Z_2 symmetry of the two space dimensions, and that the Casimir force has a maximum value which depends on the spacetime dimensions for both massless and massive cases. Especially, it is shown that the Casimir force varies as the mass of the field changes. Details of this kind of Casimir effect, will be

reviewed in Sect. 6.

Following the itinerary laid out, we review in this paper the recent developments related to scalar Casimir effect inside a p -cavity mainly based on our own works. We use the natural units $\hbar = c = k_B = 1$ in this paper.

2. The Equivalence of the Different Regularization Methods

On the physical and mathematical basic of the scalar Casimir energy in a rectangular cavity, the divergent Epstein zeta function can be reconstructed into the form of the dual convergent Epstein zeta function plus a divergent integral by repeated application of Abel-Plana formula, showing explicitly the isolation of the divergence in the zeta function scheme of regularization. Furthermore, the divergent integral can be then regulated by frequency cut-off method and interpreted as background or geometric contribution depending on different BCs. This investigation demonstrates that the zeta function regularization method is identifiable with the Abel-Plana formula approach, and it is possible that the choice of regularization methods in Casimir effect may be made for convenience.

The starting point is the energy of a massless scalar field in a rectangular cavity $\mathcal{E} = \frac{1}{2} \sum_J \omega_J$. In the case of Dirichlet or Neumann BCs,

$$\mathcal{E}^{(D/N)} = \frac{1}{2} \sum_{\vec{n} \in \mathbb{N}^D / \vec{n} \in (\mathbb{N} \cup \{\vec{0}\})^D} \sqrt{\left(\frac{\pi n_1}{L_1}\right)^2 + \left(\frac{\pi n_2}{L_2}\right)^2 + \cdots + \left(\frac{\pi n_D}{L_D}\right)^2}, \quad (2.1)$$

and in the case of periodic BCs,

$$\mathcal{E}^{(P)} = \frac{1}{2} \sum_{\vec{n} \in \mathbb{Z}^D} \sqrt{\left(\frac{2\pi n_1}{L_1}\right)^2 + \left(\frac{2\pi n_2}{L_2}\right)^2 + \cdots + \left(\frac{2\pi n_D}{L_D}\right)^2}, \quad (2.2)$$

where the superscripts “(D), (N), (P)” indicate the types of BCs. And the summation, as shown in eqs.(2.1) and (2.2), is over n_1, n_2, \dots, n_D from 1, 0 and $-\infty$ to ∞ for Dirichlet, Neumann and periodic BCs, respectively, and the prime symbol means the case $\vec{n} = \vec{0}$ has been excluded where the vector $\vec{n} = \{n_1, \dots, n_D\}$. We mostly use the periodic case as the underlying example in this section, the other two types of BCs will be briefly discussed.

2.1. Equivalence to Abel-Plana Method in One-Dimensional Case

In one-dimensional case, the reflection formula of the Riemann zeta function

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{\frac{s-1}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s), \quad (2.3)$$

which is also known as a collateral form of analytic continuation of the zeta function, plays a key role in the regularization. As eq.(2.2) reduces to

$$\mathcal{E}_1^{(P)} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \sqrt{\frac{4\pi^2 n^2}{a^2}}, \quad (2.4)$$

where a is the size of the one-dimensional box, it is quite straightforward to use eq.(2.3) to obtain the regularized finite Casimir energy

$$\mathcal{E}_1^{(P),\text{reg.}} = -\frac{1}{\pi a}\zeta(2) = -\frac{\pi}{6a}. \quad (2.5)$$

Although in the spirit of analytic continuation the ill-defined quantity is made equal to a finite one, the divergency has been implicitly removed.

To show this divergency and its removal, the regularization method using Abel-Plana formula

$$\sum_{n=1}^{\infty} u(n) = -\frac{1}{2}u(0) + \int_0^{\infty} u(x)dx + i \int_0^{\infty} \frac{u(it) - u(-it)}{e^{2\pi t} - 1} dt \quad (2.6)$$

is reviewed for comparison. With eq.(2.6) applied to eq.(2.4), the first term vanishes. The second term is $\frac{2\pi}{a} \int_0^{\infty} x^{-s} dx$, which is obviously divergent for $s < 0$. One introduces the frequency cut-off function $\exp(-\delta \frac{2\pi x}{a})$, where the parameter $\delta > 0$ has to be put $\delta = 0$ in the end, to illustrate the regularization and subtraction of this term. For $s = -1$ it becomes

$$\frac{2\pi}{a} \int_0^{\infty} x e^{-\delta \frac{2\pi x}{a}} dx = \frac{a}{2\pi\delta^2}. \quad (2.7)$$

It is proportional to the “volume” a of the one-dimensional box, and corresponds to the vacuum energy of the free unbounded space within the volume of the box. The physical Casimir energy should be the difference with respect to this kind of energy, and thus this term should be subtracted. So what is left is the third term, which is actually the integral form of a well-defined zeta function.^{79,147,148} In fact, for $s < 0$, one can carry out the integral

$$\begin{aligned} i \int_0^{\infty} \frac{(it)^{-s} - (-it)^{-s}}{e^{2\pi t} - 1} dt &= 2 \sin \frac{s\pi}{2} \int_0^{\infty} \frac{t^{-s}}{e^{2\pi t} - 1} dt \\ &= \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \zeta(1-s), \end{aligned} \quad (2.8)$$

where the integral form of Gamma function has been used. That is, the reflection formula of Riemann zeta function (2.3) is valid only after the regularization by Abel-Plana formula. Utilization of eq.(2.3) or similar analytic continuation of Riemann zeta function is actually implicit removal the vacuum energy of the free unbounded space within the volume of the one-dimensional box as the Abel-Plana formula method does explicitly.

2.2. Generalization to Higher Dimensional Cases

For a higher dimensional case, one uses the Epstein zeta function

$$Z_D(s) = \sum_{\vec{n} \in \mathbb{Z}^D \setminus \{\vec{0}\}} (\vec{n}^2)^{-\frac{s}{2}} \quad (2.9)$$

instead of the Riemann one. The reflection formula

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) Z_D(s) = \pi^{\frac{s-D}{2}} \Gamma\left(\frac{D-s}{2}\right) Z_D(D-s) \quad (2.10)$$

is also essential to the regularization of eq.(2.2). If one chooses the box to be a hypercube with the side length of a , the regularization procedure will be quite straightforward using eq.(2.10), but the removal of divergency is also hidden. Utilizing the results of one-dimensional case, the proof of eq.(2.10) from the analytic continuation aspect, the revelation of the removal of the divergency, and hence the equivalence can be presented recursively.

It is beneficial to introduce the recurrence formula of the Epstein zeta function,³⁶ which provides facilitation to the proof of eq.(2.10) and the demonstration of the equivalence between the two regularization methods at length. For homogeneous Epstein zeta function eq.(2.9), consider $Z_D(D-s)$, which is well-defined for $s < 0$,

$$\begin{aligned} Z_D(D-s) &= Z_{D-1}(D-s) + 2 \sum_{\vec{n} \in \mathbb{Z}^{D-1}} \sum_{m \in \mathbb{N}} (\vec{n}^2 + m^2)^{-\frac{D-s}{2}} \\ &= Z_{D-1}(D-s) + \frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-s}{2})} \zeta(1-s) \\ &\quad + \frac{4\pi^{\frac{D-s}{2}}}{\Gamma(\frac{D-s}{2})} \sum_{\vec{n} \in \mathbb{Z}^{D-1} \setminus \{\vec{0}\}} \sum_{m \in \mathbb{N}} \left(\frac{\sqrt{\vec{n}^2}}{m}\right)^{\frac{1-s}{2}} K_{\frac{1-s}{2}}(2\pi m \sqrt{\vec{n}^2}), \end{aligned} \quad (2.11)$$

where the Poisson summation formula

$$\sum_{n_i=-\infty}^{\infty} e^{-\frac{n_i^2 \pi^2}{L_i^2} t} = \frac{L_i}{\sqrt{\pi t}} \sum_{n_i=-\infty}^{\infty} e^{-\frac{n_i^2 L_i^2}{t}} \quad (2.12)$$

and the integral form of the modified Bessel function of the second kind $K_\nu(z)$ have been used, and the Riemann zeta term in eq.(2.11) comes from the $\vec{n} \in \{\vec{0}\}$ term. Repeat the procedure on this recurrence formula,¹⁴⁹ one arrives at

$$\begin{aligned} Z_D(D-s) &= \frac{2}{\Gamma(\frac{D-s}{2})} \sum_{j=0}^{D-1} \pi^{\frac{j}{2}} \Gamma\left(\frac{D-s-j}{2}\right) \zeta(D-s-j) \\ &\quad + \frac{4\pi^{\frac{D-s}{2}}}{\Gamma(\frac{D-s}{2})} \sum_{j=1}^{D-1} \sum_{\substack{\vec{m} \in \mathbb{N} \\ \vec{k} \in \mathbb{Z}^j \setminus \{\vec{0}\}}} \left(\frac{|\vec{k}|}{m}\right)^{\frac{D-s-j}{2}} K_{\frac{D-s-j}{2}}(2\pi m |\vec{k}|), \end{aligned} \quad (2.13)$$

which as one recalls, is well-defined for $s < 0$. With eq.(2.13) and the result of one-dimensional case, one can identify the ill-defined case of $Z_D(s)$, $s < 0$ with a finite quantity, namely prove eq.(2.10) from the analytic continuation aspect.

Comparison to the regularization using Abel-Plana formula is still helpful to explore the hidden removal of the divergency. Applying eq.(2.6) in $Z_D(s)$ (2.9),

with $s < 0$,

$$\begin{aligned}
Z_D(s) &= \sum_{\vec{n} \in \mathbb{Z}^{D-1} \setminus \{\vec{0}\}} (\vec{n}^2)^{-\frac{s}{2}} + \sum_{\substack{\vec{n} \in \mathbb{Z}^{D-1} \\ k \in \mathbb{Z} \setminus \{0\}}} (\vec{n}^2 + k^2)^{-\frac{s}{2}} \\
&= \sum_{\vec{n} \in \mathbb{Z}^{D-1} \setminus \{\vec{0}\}} (\vec{n}^2)^{-\frac{s}{2}} + 2 \sum_{\vec{n} \in \mathbb{Z}^{D-1}} \left\{ -\frac{1}{2} (\vec{n}^2)^{-\frac{s}{2}} \right. \\
&\quad \left. + \int_0^\infty (\vec{n}^2 + x^2)^{-\frac{s}{2}} dx + i \int_0^\infty \frac{(\vec{n}^2 + (it)^2)^{-\frac{s}{2}} - (\vec{n}^2 + (-it)^2)^{-\frac{s}{2}}}{e^{2\pi t} - 1} dt \right\} \\
&= 2 \int_0^\infty (x^2)^{-\frac{s}{2}} dx + 2i \int_0^\infty \frac{((it)^2)^{-\frac{s}{2}} - ((-it)^2)^{-\frac{s}{2}}}{e^{2\pi t} - 1} dt \\
&\quad + 2 \sum_{\vec{n} \in \mathbb{Z}^{D-1} \setminus \{\vec{0}\}} \int_0^\infty (\vec{n}^2 + x^2)^{-\frac{s}{2}} dx \\
&\quad + 2i \sum_{\vec{n} \in \mathbb{Z}^{D-1} \setminus \{\vec{0}\}} \int_0^\infty \frac{(\vec{n}^2 + (it)^2)^{-\frac{s}{2}} - (\vec{n}^2 + (-it)^2)^{-\frac{s}{2}}}{e^{2\pi t} - 1} dt.
\end{aligned} \tag{2.14}$$

On the RHS of the last equal sign, the first term is obviously a divergent integral, which will be canceled later. The second term is calculated in eq.(2.8). The last term is finite and since $s < 0 < 2$, can be carried out as

$$i \int_0^\infty \frac{(\vec{n}^2 + (it)^2)^{-\frac{s}{2}} - (\vec{n}^2 + (-it)^2)^{-\frac{s}{2}}}{e^{2\pi t} - 1} dt = \frac{2\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} \sum_{q \in \mathbb{N}} \left(\frac{|\vec{n}|}{q} \right)^{\frac{1-s}{2}} K_{\frac{1-s}{2}}(2q\pi|\vec{n}|). \tag{2.15}$$

The third term of eq.(2.14) is still divergent. Similar to eq.(2.14), with Abel-Plana formula (2.6) employed on the summation over \vec{n} once again, this term can be written as

$$\begin{aligned}
&2 \sum_{\vec{n} \in \mathbb{Z}^{D-1} \setminus \{\vec{0}\}} \int_0^\infty (\vec{n}^2 + x^2) dx \\
&= -2 \int_0^\infty (x^2)^{-\frac{s}{2}} dx + 4 \int_0^\infty dx \int_0^\infty dy (x^2 + y^2)^{-\frac{s}{2}} \\
&\quad + 4i \int_0^\infty dx \int_0^\infty dt \frac{(x^2 + (it)^2)^{-\frac{s}{2}} - (x^2 + (-it)^2)^{-\frac{s}{2}}}{e^{2\pi t} - 1} \\
&\quad + 4 \sum_{\vec{n} \in \mathbb{Z}^{D-2} \setminus \{\vec{0}\}} \int_0^\infty dx \int_0^\infty dy (\vec{n}^2 + x^2 + y^2)^{-\frac{s}{2}} \\
&\quad + 4i \sum_{\vec{n} \in \mathbb{Z}^{D-2} \setminus \{\vec{0}\}} \int_0^\infty dx \int_0^\infty dt \frac{(\vec{n}^2 + x^2 + (it)^2)^{-\frac{s}{2}} - (\vec{n}^2 + x^2 + (-it)^2)^{-\frac{s}{2}}}{e^{2\pi t} - 1}.
\end{aligned} \tag{2.16}$$

The two finite conjugal integrals of eq.(2.16) can also be carried out as

$$4i \int_0^\infty dx \int_0^\infty dt \frac{(x^2 + (it)^2)^{-\frac{s}{2}} - (x^2 + (-it)^2)^{-\frac{s}{2}}}{e^{2\pi t} - 1} = \frac{2\pi^{s-1}\Gamma(1 - \frac{s}{2})}{\Gamma(\frac{s}{2})} \zeta(2-s), \quad (2.17)$$

and

$$\begin{aligned} & 4i \sum_{\vec{n} \in \mathbb{Z}^{D-2} \setminus \{\vec{0}\}} \int_0^\infty dx \int_0^\infty dt \frac{(\vec{n}^2 + x^2 + (it)^2)^{-\frac{s}{2}} - (\vec{n}^2 + x^2 + (-it)^2)^{-\frac{s}{2}}}{e^{2\pi t} - 1} \\ &= \frac{4\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} \sum_{\substack{\vec{n} \in \mathbb{Z}^{D-2} \setminus \{\vec{0}\} \\ q \in \mathbb{N}}} \left(\frac{q}{|\vec{n}|}\right)^{\frac{s}{2}-1} K_{1-\frac{s}{2}}(s\pi q|\vec{n}|). \end{aligned} \quad (2.18)$$

Collecting all the pieces, one has

$$\begin{aligned} Z_D(s) &= 4 \int_0^\infty dx \int_0^\infty dy (x^2 + y^2)^{-\frac{s}{2}} + 2\pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \zeta(1-s) \\ &\quad + 4 \sum_{\vec{n} \in \mathbb{Z}^{D-2} \setminus \{\vec{0}\}} \int_0^\infty dx \int_0^\infty dy (\vec{n}^2 + x^2 + y^2)^{-\frac{s}{2}} \\ &\quad + \frac{2\pi^{s-1}\Gamma(1 - \frac{s}{2})}{\Gamma(\frac{s}{2})} \zeta(2-s) + \frac{4\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} \sum_{\vec{n} \in \mathbb{Z}^{D-2} \setminus \{\vec{0}\}} \sum_{q \in \mathbb{N}} \left(\frac{q}{|\vec{n}|}\right)^{\frac{s}{2}-1} K_{1-\frac{s}{2}}(s\pi q|\vec{n}|) \\ &\quad + \frac{2\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} \sum_{\substack{\vec{n} \in \mathbb{Z}^{D-1} \setminus \{\vec{0}\} \\ q \in \mathbb{N}}} \left(\frac{|\vec{n}|}{q}\right)^{\frac{1-s}{2}} K_{\frac{1-s}{2}}(2q\pi|\vec{n}|). \end{aligned} \quad (2.19)$$

From eq.(2.14) to eq.(2.19), we have seen the results of application of Abel-Plana formula (2.6) once and twice. Bit by bit, the divergency is put into the one-dimensional infinite integral in eq.(2.14) and then into the two-dimensional one in eq.(2.19) (the cancelation of the one-dimensional integral basically results from the BCs, different situation will be discussed later), and more and more finite conjugal integrals are isolated. Employing Abel-Plana formula on the divergent summation and repeating the procedure for another $D-2$ times, one then has

$$\begin{aligned} Z_D(s) &= 2^D \int_0^\infty (x_1^2 + x_2^2 + \cdots + x_D^2)^{-\frac{s}{2}} dx_1 dx_2 \cdots dx_D \\ &\quad + \frac{\pi^{s-\frac{D}{2}} \Gamma(\frac{D-s}{2})}{\Gamma(\frac{s}{2})} \left\{ \frac{2}{\Gamma(\frac{D-s}{2})} \sum_{j=0}^{D-1} \pi^{\frac{j}{2}} \Gamma(\frac{D-s-j}{2}) \zeta(D-s-j) \right. \\ &\quad \left. + \frac{4\pi^{\frac{D-s}{2}}}{\Gamma(\frac{D-s}{2})} \sum_{j=1}^{D-1} \sum_{\substack{\vec{n} \in \mathbb{Z}^j \setminus \{\vec{0}\} \\ q \in \mathbb{N}}} \left(\frac{|\vec{n}|}{q}\right)^{\frac{D-s-j}{2}} K_{\frac{D-s-j}{2}}(2\pi q|\vec{n}|) \right\}. \end{aligned} \quad (2.20)$$

The terms in the brace are recognized as $Z_D(D-s)$ from the eq.(2.13). So finally with the help of the Abel-Plana formula, the divergent and convergent terms of Epstein zeta function are separated as

$$Z_D(s) = 2^D \int_0^\infty (x_1^2 + x_2^2 + \cdots + x_D^2)^{-\frac{s}{2}} dx_1 dx_2 \cdots dx_D \\ + \frac{\pi^{s-\frac{D}{2}} \Gamma(\frac{D-s}{2})}{\Gamma(\frac{s}{2})} Z_D(D-s). \quad (2.21)$$

To see more clearly what this divergent part represents, the case that the side lengths $\{L_i, i = 1, \cdots, D\}$ are not necessarily equal is considered. Taking the side lengths back in eq.(2.21), for $s = -1$, the divergent part of the energy is then

$$\mathcal{E}^{(P), \text{div.}} = 2^D \pi \int_0^\infty \sqrt{(\frac{x_1}{L_1})^2 + \cdots + (\frac{x_D}{L_D})^2} dx_1 dx_2 \cdots dx_D. \quad (2.22)$$

With the frequency cut-off function similar to the one-dimensional case introduced, this term is then regulated as

$$\mathcal{E}^{(P), \text{div.}}(\delta) = 2^D \pi \int_0^\infty \sqrt{(\frac{x_1}{L_1})^2 + \cdots + (\frac{x_D}{L_D})^2} e^{-\delta \sqrt{(\frac{2\pi x_1}{L_1})^2 + \cdots + (\frac{2\pi x_D}{L_D})^2}} d^D x \\ = \frac{\Gamma(1+D)(L_1 L_2 \cdots L_D)}{2^D \pi^{\frac{D}{2}} \delta^{1+D} \Gamma(\frac{D}{2})}, \quad (2.23)$$

which is proportional to the volume of the D -dimensional box. Just like the one-dimensional case, this divergent term can be interpreted as the vacuum energy of the free unbounded space within the volume of the box.

Different BCs will give rise to divergent terms proportional to other geometric parameters. In fact, following the procedure in Ref. 44, the divergent part of the energy in the cases of Dirichlet and Neumann BCs can be expressed as

$$\mathcal{E}_{(i)}^{(D/N), \text{div.}} = (\mp \frac{1}{2})^{D-i} \frac{\pi}{2} \int_0^\infty \sqrt{(\frac{x_{\mu_1}}{L_{\mu_1}})^2 + \cdots + (\frac{x_{\mu_i}}{L_{\mu_i}})^2} d^i x, \quad (2.24)$$

where $i = 1, \cdots, D$ and $\{\mu_i\}$ is a subset of $\{1, 2, \cdots, D\}$, and the signs “ \mp ” correspond to Neumann and Dirichlet BCs, respectively. Similarly, these terms are regulated with the frequency cut-off and yield

$$\mathcal{E}_{(i)}^{(D/N), \text{div.}}(\delta) = (\mp \frac{1}{2})^{D-i} \frac{\pi}{2} \int_0^\infty \sqrt{(\frac{x_{\mu_1}}{L_{\mu_1}})^2 + \cdots + (\frac{x_{\mu_i}}{L_{\mu_i}})^2} e^{-\delta \sqrt{(\frac{\pi x_1}{L_1})^2 + \cdots + (\frac{\pi x_D}{L_D})^2}} d^i x \\ = (\mp \frac{1}{2})^{D-i} \frac{\Gamma(i+1)(L_{\mu_1} \cdots L_{\mu_i})}{2^i \pi^{\frac{i}{2}} \Gamma(\frac{i}{2}) \delta^{i+1}}. \quad (2.25)$$

The $i = D$ term is the same term obtained in the case of periodic BCs. and is considered as the vacuum energy of the free unbounded space within the volume of the box. The rest divergent terms, which are obviously proportional to the other geometric parameters of the box, are interpreted as the boundary or surface energy of the configuration. In $D = 2, 3$ cases, this is the result obtained in Ref. 3.

The physical Casimir energy should be considered as the vacuum energy with these divergent terms subtracted. When this is done, what is left in eq.(2.21) can be rearranged as

$$\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) Z_D(s) = \pi^{\frac{s-D}{2}} \Gamma(\frac{D-s}{2}) Z_D(D-s),$$

which is exactly the reflection relation of Epstein zeta function eq.(2.10). So the implicit riddance of divergency of zeta function technique is prescribed by the Abel-Plana formula method of regularization. And the two methods should be considered proven identifiable.

Through the demonstration of the equivalence of the two methods, the structure of the divergency hidden in the analytic continuation of zeta function is shown explicitly, which is also suggested by the the heat kernel expansion,^{3,4,74,75,150,151} the well appreciated and effective analysis of the divergency of zeta function.

In the light of this equivalence, together with their connection with other methods such as frequency cut-off,^{15,61,77,78,152} the consistency of using “different” methods to regularize different parts of the Casimir energy^{44,135} as we will do in Sect. 5, or to obtain different forms of the result,^{68,71,72} should not be worried about. So in the regularization of Casimir energy, any of these methods can be chosen for convenience.

3. Repulsive or Attractive Nature of the Casimir Force for Scalar Field

The question of whether the Casimir effect for a scalar field inside a rectangular cavity gives rise to an attractive or repulsive force has been discussed by many authors. In this section we will re-give the general derivation and review some investigation of this subject.

3.1. The Casimir Energy in a p -Dimensional Cavity

In Sect. 2, the configuration has been set to be a hypercube for simplicity, but in general situation, the rectangular cavities that are not closed or have unequal side lengths may have Casimir energies and forces with different signs, as considered in the literature.^{16,22,27}

3.1.1. Dirichlet and Neumann BCs

Consider the case that in eq.(2.1) only p directions have finite side lengths, namely in the rest $D-p$ directions, side lengths $L_i, i = p+1, p+2, \dots, D$ can be taken to ∞ and the summations over these n_i become integrals as

$$L_i \rightarrow \infty, \quad \frac{n_i \pi}{L_i} \rightarrow r_i, \quad \frac{\pi}{L_i} \rightarrow dr_i, \quad i = p+1, \dots, D.$$

So the energy takes the form

$$\begin{aligned}
\mathcal{E}^{(D/N)} &= \left(\prod_{i=p+1}^D L_i \right) \frac{1}{2\pi^{D-p}} \\
&\times \sum_{\vec{n} \in \mathbb{N}^p / \vec{n} \in (\mathbb{N} \cup \{\bar{0}\})^p} \int_0^\infty \left[\left(\frac{\pi n_1}{L_1} \right)^2 + \cdots + \left(\frac{\pi n_p}{L_p} \right)^2 + r_1^2 + \cdots + r_{D-p}^2 \right]^{\frac{1}{2}} d^{D-p} \mathbf{r} \\
&= \left(\prod_{i=p+1}^D L_i \right) \frac{1}{(2\sqrt{\pi})^{D-p} \Gamma(\frac{D-p}{2})} \\
&\times \sum_{\vec{n} \in \mathbb{N}^p / \vec{n} \in (\mathbb{N} \cup \{\bar{0}\})^p} \int_0^\infty r^{D-p-1} \left[\left(\frac{\pi n_1}{L_1} \right)^2 + \cdots + \left(\frac{\pi n_p}{L_p} \right)^2 + r^2 \right]^{\frac{1}{2}} dr
\end{aligned} \tag{3.1}$$

From Mellin transformation, the energy density is

$$\begin{aligned}
\varepsilon^{(D/N)} &\equiv \frac{\mathcal{E}^{(D/N)}}{\left(\prod_{i=p+1}^D L_i \right)} \\
&= -\frac{1}{2(2\sqrt{\pi})^{D-p+1}} \int_0^\infty t^{-\frac{D-p+3}{2}} \\
&\times \left(\sum_{n_1=1/n_1=0}^\infty e^{-\frac{n_1^2 \pi^2}{L_1^2} t} \right) \cdots \left(\sum_{n_p=1/n_p=0}^\infty e^{-\frac{n_p^2 \pi^2}{L_p^2} t} \right) dt \\
&= -\frac{1}{2^{D+2} \pi^{\frac{D-p+1}{2}}} \int_0^\infty t^{-\frac{D-p+3}{2}} \\
&\times \left(\sum_{n_1=-\infty}^\infty e^{-\frac{n_1^2 \pi^2}{L_1^2} t} \mp 1 \right) \cdots \left(\sum_{n_p=-\infty}^\infty e^{-\frac{n_p^2 \pi^2}{L_p^2} t} \mp 1 \right) dt.
\end{aligned} \tag{3.2}$$

Now the p different $(\sum e \mp 1)$ factors are expanded as

$$\begin{aligned}
&\left(\sum_{n_1=-\infty}^\infty e^{-\frac{n_1^2 \pi^2}{L_1^2} t} \mp 1 \right) \cdots \left(\sum_{n_p=-\infty}^\infty e^{-\frac{n_p^2 \pi^2}{L_p^2} t} \mp 1 \right) \\
&= \sum_{q=0}^{p-1} (\mp 1)^q \sum_{\{i_1, \dots, i_{p-q}\} \in \{1, 2, \dots, p\}} \left(\sum_{n_{i_1}=-\infty}^\infty e^{-\frac{n_{i_1}^2 \pi^2}{L_{i_1}^2} t} \right) \cdots \left(\sum_{n_{i_{p-q}}=-\infty}^\infty e^{-\frac{n_{i_{p-q}}^2 \pi^2}{L_{i_{p-q}}^2} t} \right),
\end{aligned} \tag{3.3}$$

where the summation $\sum_{\{i_1, \dots, i_{p-q}\} \in \{1, 2, \dots, p\}}$ means over all the $(p-q)$ -element subsets $\{i_1, \dots, i_{p-q}\}$ of the set $\{1, 2, \dots, p\}$. Note that if all L_i are equal, eq.(3.3)

is just the binary expansion

$$\left(\sum_{n=-\infty}^{\infty} e^{-\frac{n^2 \pi^2}{L^2} t} \mp 1 \right)^p = \sum_{q=0}^{p-1} (\mp 1)^q C_p^q \left(\sum_{n=-\infty}^{\infty} e^{-\frac{n^2 \pi^2}{L^2} t} \right)^{p-q}.$$

Poisson summation eq.(2.12) is the essential step of the regularization, which can be applied to all the summations. Taking eqs.(2.12) and (3.3) back into eq.(3.2) one then has

$$\begin{aligned} \varepsilon^{(D/N), \text{reg.}} &= - \frac{1}{2^{D+2} \pi^{\frac{D-p+1}{2}}} \int_0^\infty t^{-\frac{D-p+3}{2}} \\ &\quad \times \left(\sum_{n_1=-\infty}^{\infty} e^{-\frac{n_1^2 \pi^2}{L_1^2} t} \mp 1 \right) \cdots \left(\sum_{n_p=-\infty}^{\infty} e^{-\frac{n_p^2 \pi^2}{L_p^2} t} \mp 1 \right) dt \\ &= - \frac{1}{2^{D+2} \pi^{\frac{D-p+1}{2}}} \sum_{q=0}^{p-1} (\mp 1)^q \sum_{\{i_1, \dots, i_{p-q}\} \in \{1, 2, \dots, p\}} \frac{L_{i_1} \cdots L_{i_{p-q}}}{\pi^{\frac{p-q}{2}}} \\ &\quad \times \sum_{\vec{n} \in \mathbb{Z}^{p-q}} \int_0^\infty t^{-\frac{D-q+3}{2}} \exp \left[-\frac{n_{i_1}^2 L_{i_1}^2 + \cdots + n_{i_{p-q}}^2 L_{i_{p-q}}^2}{t} \right] dt \\ &= - \frac{1}{2^{D+2}} \sum_{q=0}^{p-1} \frac{(\mp 1)^q \Gamma(\frac{D-q+1}{2})}{\pi^{\frac{D-q+1}{2}}} \\ &\quad \times \sum_{\{i_1, \dots, i_{p-q}\} \in \{1, 2, \dots, p\}} L_{i_1} \cdots L_{i_{p-q}} Z_{p-q}(L_{i_1}, \dots, L_{i_{p-q}}; D-q+1), \end{aligned} \quad (3.4)$$

where the Epstein zeta function generalized from eq.(2.9) is defined as

$$Z_k(a_1, a_2, \dots, a_k; s) \equiv \sum_{\vec{n} \in \mathbb{Z}^k \setminus \{\vec{0}\}} (a_1^2 n_1^2 + \cdots + a_k^2 n_k^2)^{-\frac{s}{2}}.$$

Eq.(3.4) is the general form of the regularized Casimir energy (density) in a rectangular cavity with Dirichlet or Neumann BCs. The “−” sign in $(\mp 1)^q$ corresponds to Dirichlet BCs and “+” sign to Neumann BCs.

In two-dimensional closed box with Dirichlet BCs, i.e. $D = p = 2$, let $L_1 = a$, $L_2 = b$, eq.(3.4) is

$$\mathcal{E}_2^{(D), \text{reg.}} = - \frac{ab}{32\pi} Z_2(a, b; 3) + \frac{\pi}{48} \left(\frac{1}{a} + \frac{1}{b} \right). \quad (3.5)$$

And in three-dimensional closed box with Dirichlet BCs, i.e. $D = p = 3$, let $L_1 = a$, $L_2 = b$, $L_3 = c$, eq.(3.4) is

$$\begin{aligned} \mathcal{E}_3^{(D), \text{reg.}} &= - \frac{abc}{32\pi^2} Z_3(a, b, c; 4) + \frac{bc}{64\pi} Z_2(b, c; 3) + \frac{ac}{64\pi} Z_2(a, c; 3) \\ &\quad + \frac{ab}{64\pi} Z_2(a, b; 3) - \frac{\pi}{96} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right). \end{aligned} \quad (3.6)$$

Both eqs.(3.5) and (3.6) have been obtained in Refs. 18, 20 and reviewed in Ref. 3.

In the case that the cavity has equal edges $L_i = L, i = 1, \dots, p$ with Dirichlet BCs, eq.(3.4) becomes

$$\varepsilon^{(D),\text{reg.}} = -\frac{L^{p-D-1}}{2^{D+2}} \sum_{q=0}^{p-1} C_p^q \frac{(\mp 1)^q \Gamma(\frac{D-q+1}{2})}{\pi^{\frac{D-q+1}{2}}} Z_{p-q}(1, \dots, 1; D-q+1), \quad (3.7)$$

which is what Caruso et al. have obtained in Ref. 16.

Taking “+” in every “ \mp ” sign, the energy (density) in Neumann case is always negative. However, since there is a factor of $(-1)^q$ in eq.(3.4) for Dirichlet BCs, the sign of the energy (density) is not yet determinative. In Ref. 16 the authors analysed the case in which all the edges are equal. If p is odd, it can be seen analytically that the energy (density) is also negative. If p is even, it is found out numerically that for every even p there is a critical $D = D_c$, and the energy (density) is positive for $D < D_c$.

As for the more general case with unequal sidelengths, Ref. 22 provides an angle to address this issue. According to definition, the energy density in p -cavity $\varepsilon_p^{(D)}$ with Dirichlet BCs has

$$\mathcal{E}^{(D)} = \left(\prod_{i=p+1}^D L_i \right) \varepsilon_p^{(D)}, \text{ for } L_{p+1}, \dots, L_D \gg L_1, \dots, L_p. \quad (3.8)$$

It follows that

$$\varepsilon_p^{(D)} = L_{p-q+1} L_{p-q+2} \dots L_p \varepsilon_{p-q}^{(D)}, \text{ for } L_{p-q+1}, \dots, L_p \gg L_1, \dots, L_{p-q}. \quad (3.9)$$

Eqs.(3.8) and (3.9) are also valid for regularized energy densities. This shows that the ratio of side lengths may have a critical value for energy density to change sign. For example, $\varepsilon_2^{(D),\text{reg.}} = L_2 \varepsilon_1^{(D),\text{reg.}}$ for $L_2 \gg L_1$. Now $\varepsilon_1^{(D),\text{reg.}}$ is always negative and $\varepsilon_2^{(D),\text{reg.}}(L_1 = L_2)$ is positive for $D < D_{\text{crit}} = 6$.¹⁶ Since $\varepsilon_2^{(D),\text{reg.}}(L_1, L_2)$ is a continuous function for $L_2 > 0$, there thus exists a critical ratio $\mu = L_2/L_1 = \mu_{\text{crit}}$ for $\varepsilon_2^{(D),\text{reg.}}$ to turn from positive to negative. Furthermore, there is a Z_2 symmetry $L_1 \leftrightarrow L_2$ for energy function. Numerical calculations²² show that for $p = 2$ and $D < D_{\text{crit}} = 6$,¹⁶ the critical ratio μ_{crit} does exist. When $D = 2 = p$, $\mu_{\text{crit}} = 2.737$, and as D increases μ_{crit} becomes smaller. And one can conclude that if $L_1/L_2 > 2.737$ or $L_2/L_1 > 2.737$, the energy density $\varepsilon_p^{(D,\text{reg.})} < 0$ for any space dimensionality D .

3.1.2. Periodic BCs

In periodic case eq.(2.2), $L_i, i = p+1, \dots, D$ are also taken to ∞ and summations over these n_i are turned into integrals with

$$L_i \rightarrow \infty, \quad \frac{2n_i\pi}{L_i} \rightarrow r_i, \quad \frac{2\pi}{L_i} \rightarrow dr_i, \quad i = p+1, \dots, D.$$

And then similarly, energy density is

$$\begin{aligned}
\varepsilon^{(P)} &\equiv \frac{\mathcal{E}^{(P)}}{\left(\prod_{i=p+1}^D L_i\right)} \\
&= \frac{1}{2^{D-p}\pi^{\frac{D-p}{2}}\Gamma(\frac{D-p}{2})} \sum_{\vec{n} \in \mathbb{Z}^p} \int_0^\infty r^{D-p-1} \left[\left(\frac{2\pi n_1}{L_1}\right)^2 + \dots + \left(\frac{2\pi n_p}{L_p}\right)^2 + r^2 \right]^{\frac{1}{2}} dr \\
&= -\frac{1}{2(2\sqrt{\pi})^{D-p+1}} \int_0^\infty t^{-\frac{D-p+3}{2}} \left(\sum_{n_1=-\infty}^\infty e^{-4\frac{n_1^2\pi^2}{L_1^2}t} \right) \dots \left(\sum_{n_p=-\infty}^\infty e^{-4\frac{n_p^2\pi^2}{L_p^2}t} \right) dt.
\end{aligned} \tag{3.10}$$

Again, the Poisson summation eq.(2.12) is employed for regularization,

$$\begin{aligned}
\varepsilon^{(P),\text{reg.}} &= -\frac{1}{2(2\sqrt{\pi})^{D-p+1}} \int_0^\infty t^{-\frac{D-p+3}{2}} \\
&\quad \times \left(\frac{L_1}{2\sqrt{\pi t}} \sum_{n_1=-\infty}^\infty e^{-\frac{n_1^2 L_1^2}{4t}} \right) \dots \left(\frac{L_p}{2\sqrt{\pi t}} \sum_{n_p=-\infty}^\infty e^{-\frac{n_p^2 L_p^2}{4t}} \right) dt \\
&= -\frac{L_1 \cdots L_p}{2(2\sqrt{\pi})^{D+1}} \sum_{\vec{n} \in \mathbb{Z}^p} \int_0^\infty t^{-\frac{D+3}{2}} e^{-(n_1^2 L_1^2 + \dots + n_p^2 L_p^2) \frac{1}{4t}} dt \\
&= -\frac{L_1 \cdots L_p}{2\pi^{\frac{D+1}{2}}} \Gamma\left(\frac{D+1}{2}\right) Z_p(L_1, \dots, L_p; D+1).
\end{aligned} \tag{3.11}$$

Eq.(3.11) is the general form of the regularized Casimir energy (density) in a rectangular cavity with periodic BCs, which is in agreement of what Ambjørn and Wolfram have obtained,¹² and obvious is always negative. In two and three-dimensional closed box with periodic BCs, i.e. $D = p = 2, 3$, eq.(3.11) recovers the results obtained in Refs. 18, 20 and reviewed in Ref. 3.

As a summary, we have put all the results reviewed above in Table 1.

Table 1. Sign of the Casimir energy density of a massless scalar field confined in p -cavity of $(D + 1)$ -dimensional spacetime

$\varepsilon^{\text{reg.}}$	equal p edges		unequal p edges
	p odd	p even	
periodic			< 0
Neumann			< 0
Dirichlet	< 0	> 0 for $D < D_c$	depends on p, D and the ratios of sidelengths
		< 0 for $D > D_c$	

3.2. The Sign of the Force

From eqs.(3.4) and (3.11), one can calculate the Casimir force per unit area for a specific p . For $p = 1$,

$$\varepsilon_1^{(D/N),\text{reg.}} = -\frac{\Gamma(\frac{D+1}{2})\zeta(D+1)}{2^{D+1}\pi^{\frac{D+1}{2}}L^D}, \quad (3.12)$$

and $\varepsilon_1^{(P),\text{reg.}} = 2^{D+1}\varepsilon_1^{(D/N),\text{reg.}}$. So the Casimir force density for $p = 1$ is always negative, namely attractive for all three types of BCs.

In the $p = 2$ case, one can expand the Epstein zeta function $Z_2(L_1, L_2; D+1)$ in the similar manner as eq.(2.13) and rewrite the energy density eq.(3.4) for Dirichlet BCs as

$$\begin{aligned} \varepsilon_2^{(D),\text{reg.}} = & -\frac{1}{2^{D-1}L_1^{\frac{D}{2}}L_2^{\frac{D}{2}-1}} \sum_{n_1, n_2=1}^{\infty} \left(\frac{n_1}{n_2}\right)^{\frac{D}{2}} K_{\frac{D}{2}}\left(2n_1n_2\pi\frac{L_2}{L_1}\right) \\ & -\frac{\zeta(D+1)\Gamma(\frac{D+1}{2})L_2}{2^{D+1}\pi^{\frac{D+1}{2}}L_1^D} + \frac{\zeta(D)\Gamma(\frac{D}{2})}{2^{D+1}\pi^{\frac{D}{2}}L_1^{D-1}}. \end{aligned} \quad (3.13)$$

It follows that the force density along the direction of L_2 is

$$\begin{aligned} -\frac{\partial\varepsilon_2^{(D),\text{reg.}}}{\partial L_2} = & \frac{(1-\frac{D}{2})}{2^{D-1}(L_1L_2)^{\frac{D}{2}}} \sum_{n_1, n_2=1}^{\infty} \left(\frac{n_1}{n_2}\right)^{\frac{D}{2}} K_{\frac{D}{2}}\left(2n_1n_2\pi\frac{L_2}{L_1}\right) \\ & + \frac{\pi}{2^{D-2}L_1^{\frac{D}{2}+1}L_2^{\frac{D}{2}-1}} \sum_{n_1, n_2=1}^{\infty} \frac{n_1^{\frac{D}{2}+1}}{n_2^{\frac{D}{2}-1}} K'_{\frac{D}{2}}\left(2n_1n_2\pi\frac{L_2}{L_1}\right) \\ & + \frac{\zeta(D+1)\Gamma(\frac{D+1}{2})}{2^{D+1}\pi^{\frac{D+1}{2}}L_1^D}, \end{aligned} \quad (3.14)$$

where $K'_\nu(z) \equiv \frac{\partial K_\nu(z)}{\partial z}$. The first two terms in eq.(3.14) are negative and monotonically increasing functions with L_2 , while the last term is positive and is independent of L_2 . So it is expected that the Casimir energy density has a maximum with respect to L_2 , and that the Casimir force density will turn from attractive to repulsive when L_2 increases. Table 2²² shows the maximum of the energy density for $p = 2$ and various D with Dirichlet BCs and the corresponding ratios of side lengths μ_{\max} , as well as the energy density at $L_1 = L_2$ for contrast.

Similar analyses can be applied to eq.(3.4) of Neumann case and eq.(3.11) of periodic case. It is found that although the energy densities with these two types of BCs is always negative, the force densities show similar behaviors as in Dirichlet case, i.e. they turn from attractive to repulsive as L_2 increases.

For higher dimensional case, if one considers the force along only the direction of L_p , namely $-\partial\varepsilon_p^{\text{reg.}}/\partial L_p$ and let all the rest side lengths be equal $L_1 = L_2 = \dots = L_{p-1}$, the analyses of 2-dimensional case above can be extended straightforward and the conclusion is still valid.

Table 2. The maximum value of the Casimir energy densities at $L_2/L_1 = \mu_{\max}$ for massless scalar fields satisfying Diriclet BCs inside a cavity with unequal edges in a $(D + 1)$ -dimensional spacetime, where L_1 is the chosen unit length. Meantime, the values of $\varepsilon_2^{(D),\text{reg.}}$ at $L_1 = L_2$ are listed to contrast with $\varepsilon_{2,\max}^{(D),\text{reg.}}$.

D	μ_{\max}	$\varepsilon_{2,\max}^{(D),\text{reg.}}$	$\varepsilon_2^{(D),\text{reg.}}(L_1 = L_2)$
5	$1 + (1 \times 10^{-7})$	0.0001146407	0.0001146408
6	1.0102	-0.0000192394	-0.0000194771
7	1.0375	-0.0000366757	-0.0000386962
8	1.0575	-0.0000311072	-0.0000341599
9	1.0724	-0.0000231299	-0.0000263762
10	1.0830	-0.0000167097	-0.0000197328
11	1.0911	-0.0000121189	-0.0000147795
12	1.0968	-0.0000089401	-0.0000112286
13	1.1008	-0.0000067468	-0.0000087042
14	1.1034	-0.0000052212	-0.0000069027
15	1.1049	-0.0000041471	-0.0000056059
16	1.1058	-0.0000033808	-0.0000043828
17	1.1063	-0.0000028276	-0.0000039731
18	1.1065	-0.0000024248	-0.0000034650
19	1.1067	-0.0000021304	-0.0000030916

These results permit us to discuss a possible application for the Abraham-Lorentz electron model.²² A $p = 3$ rectangular cavity with walls of perfect conductivity is considered. The electromagnetic field satisfies the BCs $\mathbf{n} \cdot \mathbf{B} = 0$ and $\mathbf{n} \cdot \mathbf{E} = 0$. The Casimir energy $\mathcal{E}^{(\text{em})}(L_1, L_2, L_3)$ of the electromagnetic field can be written in terms of the massless scalar field as

$$\mathcal{E}^{(\text{em})}(L_1, L_2, L_3) = 2\mathcal{E}^{(D)}(L_1, L_2, L_3) + \mathcal{E}^{(D)}(L_1, L_2) + \mathcal{E}^{(D)}(L_1, L_3) + \mathcal{E}^{(D)}(L_2, L_3). \quad (3.15)$$

Use will also be made of the well-known fact¹⁵³ that the order of magnitude of the electromagnetic zero-point energy does not change if one deforms a spherical shell of radius a into a cubic shell of length, with $L \approx 2a$. On the other hand, the Abraham-Lorentz model describes the electron as a conducting spherical shell of radius a . To guarantee the stability of the electron Poincaré stresses had to be postulated. Casimir¹⁵⁴ proposed to extend the classical electron model by taking into account the zero-point fluctuations of the electromagnetic field inside and outside of the conducting shell. Unfortunately, the Casimir model of the electron fails, at least in the $L_1 = L_2 = L_3 \approx 2a$ case, because the Casimir energy of an S^2 electron is positive from eq.(3.15). Does this argument still hold for rectangular cavity? The answer is no, and it can be shown that the zero-point energy is negative when we choose lengths of edges, appropriately. We take, for example $L_1 = 1.6$ and $L_2 = L_3 = 1$, then $\mathcal{E}^{(\text{em})}(L_1, L_2, L_3) \approx -2 \times 10^{-3}$. Therefore, Casimir-like model of electron could be stable. Note that, in this case, the condition of stability will be satisfied only for a particular shape and size.

4. Casimir Pistons

The rectangular Casimir piston is a variant of the Casimir effect in a rectangular box, where the box is divided by a movable partition. Based on the Casimir energy reviewed in Sect. 3, for simplicity, we will focus on the configuration of a p -dimensional (note that the dimensionality of the space $D = p$) rectangular piston where the length of the variable edges are a and $L - a$ while the rest sidelengths are all b . Fig.1 illustrates a three-dimensional piston. It follows that the Casimir force

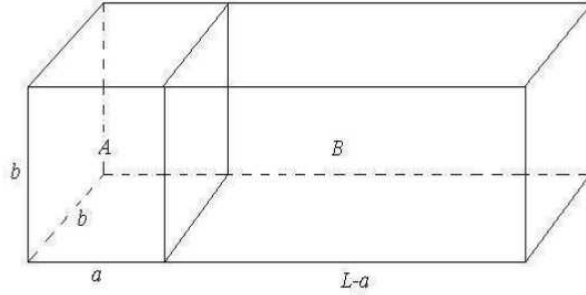


Fig. 1. Three-dimensional piston with the variable edges of a and $L - a$ and fixed edges of b .

on the piston is

$$\mathcal{F}_p = -\frac{\partial}{\partial a} [\mathcal{E}_p(A) + \mathcal{E}_p(B) + \mathcal{E}^{\text{out}}] = -\frac{\partial}{\partial a} [\mathcal{E}_p(A) + \mathcal{E}_p(B)], \quad (4.1)$$

where $\mathcal{E}_p(A)$ and $\mathcal{E}_p(B)$ are the Casimir energy in compartment A and B indicated in Fig.1, respectively, and \mathcal{E}^{out} is the energy outside the box, which provides no contribution to the force and will be neglected in the following.

4.1. Massless Scalar Field with Dirichlet BCs

For Dirichlet BCs, the energy of a massless scalar field in a rectangular cavity is given in eq.(3.4). The case that $D = p$ and with “ $-$ ” sign in “ \mp ” is retained here:

$$\begin{aligned} \mathcal{E}_p^{(D), \text{reg.}} = & -\frac{1}{2^{p+2}} \sum_{q=0}^{p-1} \frac{(-1)^q \Gamma(\frac{p-q+1}{2})}{\pi^{\frac{p-q+1}{2}}} \\ & \times \sum_{\{i_1, \dots, i_{p-q}\} \in \{1, 2, \dots, p\}} L_{i_1} \cdots L_{i_{p-q}} Z_{p-q}(L_{i_1}, \dots, L_{i_{p-q}}; p - q + 1). \end{aligned} \quad (4.2)$$

For compartment A, the energy is expressed as

$$\begin{aligned} \mathcal{E}_p^{(\text{D}),\text{reg.}}(a, b) = & -\frac{a}{2^{p+2}} \sum_{q=0}^{p-1} \frac{(-1)^q \Gamma(\frac{p-q+1}{2})}{\pi^{\frac{p-q+1}{2}}} \\ & \times C_{p-1}^q b^{p-q-1} \sum_{(j, \vec{k}) \in \mathbb{Z}^{p-q}} ' \left[a^2 j^2 + b^2 \vec{k}^2 \right]^{-\frac{p-q+1}{2}} + c(b), \end{aligned} \quad (4.3)$$

where similarly \vec{k} represents $\{k_1, \dots, k_{p-1}\}$ and the last term $c(b)$ contains terms independent of a and thus provides no contribution to the force on the piston.

For $p = 1$, the energy is simply

$$\mathcal{E}_1^{(\text{D}),\text{reg.}}(a) = -\frac{\zeta(2)}{4\pi a} = -\frac{\pi}{24a}. \quad (4.4)$$

Thus the force on the piston is

$$\mathcal{F}_1^{(\text{D})}(a) = -\lim_{L \rightarrow \infty} \frac{\partial}{\partial a} \left[\mathcal{E}_1^{(\text{D})}(a) + \mathcal{E}_1^{(\text{D})}(L - a) \right] = -\frac{\pi}{24a^2}. \quad (4.5)$$

For $p \geq 2$, eq.(4.3) is then manipulated as follows

$$\begin{aligned} \mathcal{E}_p^{(\text{D}),\text{reg.}}(a, b) = & -\frac{a}{2^{p+2}} \sum_{q=0}^{p-2} \frac{(-1)^q \Gamma(\frac{p-q+1}{2})}{\pi^{\frac{p-q+1}{2}}} C_{p-1}^q b^{p-q-1} \\ & \times \sum_{j=-\infty}^{\infty} ' \sum_{\vec{k} \in \mathbb{Z}^{p-q-1}} \left[a^2 j^2 + b^2 \vec{k}^2 \right]^{-\frac{p-q+1}{2}} \\ & - \frac{a}{2^{p+2} b^2} \sum_{q=0}^{p-2} (-1)^q C_{p-1}^q Z_{p-q-1}(p-q+1) + \frac{(-1)^p}{2^{p+1} \pi a} \zeta(2) + c(b) \\ = & -\frac{1}{2^p b} \sum_{q=0}^{p-2} (-1)^q C_{p-1}^q \sum_{j=1}^{\infty} \sum_{\vec{k} \in \mathbb{Z}^{p-q-1}} ' \frac{|\vec{k}|}{j} K_1\left(2\frac{a}{b} j |\vec{k}| \pi\right) \\ & - \frac{a}{2^{p+2} b^2} \sum_{q=0}^{p-2} (-1)^q C_{p-1}^q Z_{p-q-1}(p-q+1) + c(b), \end{aligned} \quad (4.6)$$

where the Poisson summation eq.(2.12) and the integral form of the modified Bessel function of the second kind have come to one's aid, and the homogeneous Epstein zeta terms are the $j = 0$ terms.

Now the force on the piston is

$$\begin{aligned} \mathcal{F}_p^{(\text{D})} = & -\lim_{L \rightarrow \infty} \frac{\partial}{\partial a} \left[\mathcal{E}_p^{(\text{D}),\text{reg.}}(a, b) + \mathcal{E}_p^{(\text{D}),\text{reg.}}(L - a, b) \right] \\ = & \frac{\pi}{2^{p-1} b^2} \sum_{q=0}^{p-2} (-1)^q C_{p-1}^q \sum_{j=1}^{\infty} \sum_{\vec{k} \in \mathbb{Z}^{p-q-1}} ' |\vec{k}|^2 K_1'\left(2\frac{a}{b} j |\vec{k}| \pi\right), \end{aligned} \quad (4.7)$$

where $K_1'(x) = \frac{\partial}{\partial x} K_1(x)$.

When $p = 2$,

$$\mathcal{F}_2^{(D)} = \frac{\pi}{b^2} \sum_{j,k=1}^{\infty} k^2 K_1' \left(2 \frac{a}{b} j k \pi \right), \quad (4.8)$$

which is the result obtained by Cavalcanti.⁶⁷

4.2. Massive Scalar Field with Dirichlet BCs

On the other hand, the Casimir effect for the massive scalar field also studied by some authors.^{4, 85, 155} As is known that the Casimir effect vanishes as the mass m of the field goes to infinity since there are no more quantum fluctuations in the limit. We review here the study of the precise way the Casimir energy varies as the mass changes. Since the regularized energy for massive case with mass m is not given in previous sections, in order to be self-contained, we start from the unregularized energy in compartment A:

$$\mathcal{E}_p^{(D)}(a, b, m) = \frac{1}{2} \sum_{j=1}^{\infty} \sum_{k_1, \dots, k_{p-1}=1}^{\infty} \sqrt{\frac{\pi^2 j^2}{a^2} + \frac{\pi^2 \vec{k}^2}{b^2} + m^2}. \quad (4.9)$$

The regularization procedure is similar, the Mellin transformation and Poisson summation eq.(2.12) are employed on the summations.

$$\begin{aligned} \mathcal{E}_p^{(D)}(a, b, m) &= -\frac{1}{4\sqrt{\pi}} \sum_{j=1}^{\infty} \sum_{k_1, \dots, k_{p-1}=1}^{\infty} \int_0^{\infty} t^{-\frac{3}{2}} e^{-\left[\frac{j^2 \pi^2}{a^2} + \frac{\vec{k}^2 \pi^2}{b^2} + m^2\right]t} dt \\ &= -\frac{1}{2^{p+2}\sqrt{\pi}} \int_0^{\infty} t^{-\frac{3}{2}} \left(\sum_{j=-\infty}^{\infty} e^{-\frac{j^2 \pi^2}{a^2} t} - 1 \right) \\ &\quad \times \left(\sum_{k=-\infty}^{\infty} e^{-\frac{\vec{k}^2 \pi^2}{b^2} t} - 1 \right)^{p-1} e^{-m^2 t} dt \\ &= -\frac{1}{2^{p+2}\sqrt{\pi}} \sum_{q=0}^{p-1} (-1)^q C_{p-1}^q \int_0^{\infty} t^{-\frac{3}{2}} e^{-m^2 t} \\ &\quad \times \left(\sum_{j=-\infty}^{\infty} e^{-\frac{j^2 \pi^2}{a^2} t} \right) \left(\sum_{\vec{k} \in \mathbb{Z}^{p-q-1}} e^{-\frac{\vec{k}^2 \pi^2}{b^2} t} \right) dt + c(b), \end{aligned} \quad (4.10)$$

where all the a independent terms are still put in $c(b)$, then the regularized energy

$$\begin{aligned}
& \mathcal{E}_p^{(D), \text{reg.}}(a, b, m) \\
&= -\frac{a}{2^p} \sum_{q=0}^{p-2} (-1)^q C_{p-1}^q \frac{b^{p-q-1}}{\pi^{\frac{p-q+1}{2}}} \sum_{j=1}^{\infty} \sum_{\vec{k} \in \mathbb{Z}^{p-q-1}} \left(\frac{m^2}{j^2 a^2 + \vec{k}^2 b^2} \right)^{\frac{p-q+1}{4}} \\
&\quad \times K_{\frac{p-q+1}{2}} \left(2m \sqrt{j^2 a^2 + \vec{k}^2 b^2} \right) \\
&\quad - \frac{a}{2^{p+1}} \sum_{q=0}^{p-2} (-1)^q C_{p-1}^q \frac{b^{p-q-1}}{\pi^{\frac{p-q+1}{2}}} \sum_{\vec{k} \in \mathbb{Z}^{p-q-1}} \left(\frac{m^2}{\vec{k}^2 b^2} \right)^{\frac{p-q+1}{4}} K_{\frac{p-q+1}{2}} \left(2m |\vec{k}| b \right) \\
&\quad + \frac{(-1)^p}{2^p \pi} \sum_{j=1}^{\infty} \frac{m}{j} K_1(2mja) + \frac{(-1)^p a \Gamma(-1) m^2}{2^{p+2} \pi} + c(b).
\end{aligned} \tag{4.11}$$

And the force yields

$$\begin{aligned}
\mathcal{F}_p^{(D)}(a, b, m) &= -\frac{1}{2^p} \sum_{q=0}^{p-2} (-1)^q C_{p-1}^q \frac{b^{p-q-1}}{\pi^{\frac{p-q+1}{2}}} \\
&\quad \times \sum_{j=1}^{\infty} \sum_{\vec{k} \in \mathbb{Z}^{p-q-1}} \left[\frac{m^{\frac{p-q+1}{2}}}{\left(a^2 j^2 + b^2 \vec{k}^2 \right)^{\frac{p-q+1}{4}}} K_{\frac{p-q+1}{2}} \left(2\sqrt{a^2 j^2 + b^2 \vec{k}^2} m \right) \right. \\
&\quad \left. - \frac{m^{\frac{p-q+3}{2}}}{\left(a^2 j^2 + b^2 \vec{k}^2 \right)^{\frac{p-q+3}{4}}} K_{\frac{p-q+3}{2}} \left(2\sqrt{a^2 j^2 + b^2 \vec{k}^2} m \right) \right] \\
&\quad - \frac{(-1)^p}{2^p \pi a} \sum_{j=1}^{\infty} \frac{m}{j} K_1(2mja) + \frac{(-1)^p m^2}{2^{p-1} \pi a} \sum_{j=1}^{\infty} K_2(2mja).
\end{aligned} \tag{4.12}$$

For $p = 2$, eqs.(4.11) and (4.12) give the results obtained in Ref. 40, in which the authors also consider the influence of the mass on the force illustrated in Figs.2 and 3.

4.3. Hybrid BCs

The same configuration as Fig. 1 indicates is considered, except that the BC on the piston is Neumann while those on the rest are Dirichlet, which is dubbed as the hybrid BCs and denoted in the superscript “(H)”.

The energy in compartment A with hybrid BCs, before regularization, is

$$\mathcal{E}_p^{(H)}(a, b) = \frac{1}{2} \sum_{j=0}^{\infty} \sum_{k_1, \dots, k_{p-1}=1}^{\infty} \sqrt{\left(j + \frac{1}{2} \right)^2 \frac{\pi^2}{a^2} + \frac{\vec{k}^2 b^2}{b^2}}, \tag{4.13}$$

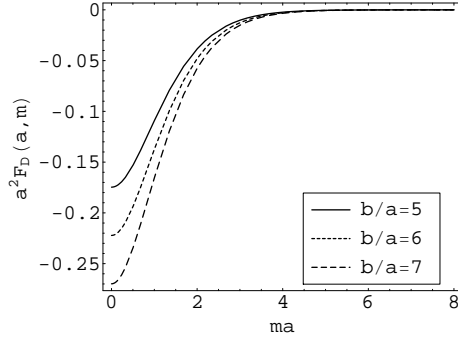


Fig. 2. The Casimir force on the piston (in units of $\frac{1}{a^2}$) versus ma for different ratio of b/a with Dirichlet BCs.

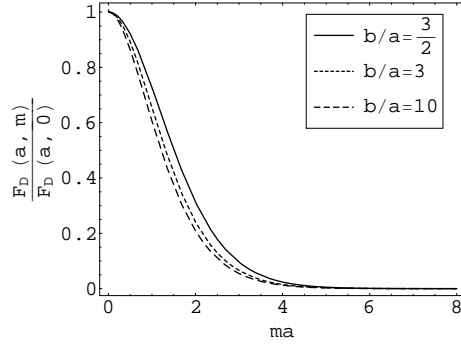


Fig. 3. The ratio of the Casimir force on the piston for a massive scalar field and massless scalar field versus ma for different ratio of b/a with Dirichlet BCs.

for massless case and

$$\mathcal{E}_p^{(H)}(a, b, m) = \frac{1}{2} \sum_{j=0}^{\infty} \sum_{k_1, \dots, k_{p-1}=1}^{\infty} \sqrt{\left(j + \frac{1}{2}\right)^2 \frac{\pi^2}{a^2} + \frac{\vec{k}^2 b^2}{b^2} + m^2}, \quad (4.14)$$

for massive case. One can re-express eqs.(4.13) and (4.14) as³⁹

$$\mathcal{E}_p^{(H)}(a, b) = \mathcal{E}_p^{(D)}(2a, b) - \mathcal{E}_p^{(D)}(a, b), \quad (4.15)$$

$$\mathcal{E}_p^{(H)}(a, b, m) = \mathcal{E}_p^{(D)}(2a, b, m) - \mathcal{E}_p^{(D)}(a, b, m). \quad (4.16)$$

Therefore, the force on the piston with hybrid BCs can be obtained by

$$\mathcal{F}_p^{(H)}(a, b) = 2\mathcal{F}_p^{(D)}(2a, b) - \mathcal{F}_p^{(D)}(a, b), \quad (4.17)$$

$$\mathcal{F}_p^{(H)}(a, b, m) = 2\mathcal{F}_p^{(D)}(2a, b, m) - \mathcal{F}_p^{(D)}(a, b, m). \quad (4.18)$$

For $p = 1$ and massless case, from eq.(4.5),

$$\mathcal{F}_1^{(H)}(a) = 2\mathcal{F}_1^{(D)}(2a) - \mathcal{F}_1^{(D)}(a) = \frac{\pi}{48a^2}, \quad (4.19)$$

which was obtained in Ref. 39, and can also be obtained by use of exponential cutoff technique.⁸⁹

For $p \geq 2$, from eqs.(4.7) one can obtain

$$\mathcal{F}_p^{(H)}(a, b) = \frac{\pi}{2^{p-1}b^2} \sum_{q=0}^{p-2} (-1)^q C_{p-1}^q \sum_{j=1}^{\infty} \sum_{\vec{k} \in \mathbb{Z}^{p-q-1}} '|\vec{k}|^2 \left[2K_1' \left(4\frac{a}{b} j |\vec{k}| \pi \right) - K_1' \left(2\frac{a}{b} j |\vec{k}| \pi \right) \right], \quad (4.20)$$

for massless case, and for $p = 2, 3$, it is the result obtained in Ref. 39, in which numerical computation has been carried out for all $p = 1, 2, 3$ cases showing that the force is always repulsive (see Fig.4 for $p = 3$), in contrast with the same problem where the BCs are Dirichlet on all surfaces. For the massive case, the influence of the mass is similar to the Dirichlet BCs (see Ref. 39).

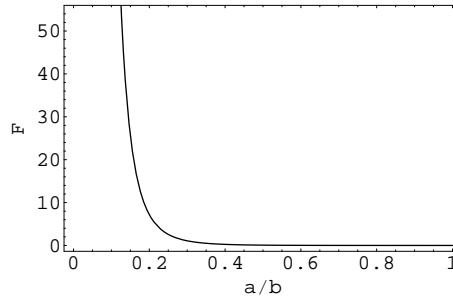


Fig. 4. Casimir force \mathcal{F} (in units hc/b^2) on a three-dimensional piston versus a/b where a is the plate separation and b is the length of the sides of the square base.

The problem of hybrid BCs reviewed here is analogous to the problem in electromagnetic field that the piston is an infinitely permeable plate and the other sides of the cavity are perfectly conducting ones. This problem may be connected with the study of dynamical Casimir effect and may be applied to the fabrication of microelectromechanical system (MEMS).

5. Nonzero Temperature Casimir Effect

In previous sections, the Casimir effect of a quantum field on a vacuum state has been reviewed, we now turn to the field on thermal equilibrium states characterized by a finite temperature T .

5.1. Two Parts of the Free Energy

In quantum field theory, the imaginary-time Matsubara formalism may be the easiest way to introduce the influence of temperature to a system. In this formalism

one makes the time coordinate rotate as $t \rightarrow -i\tau$ and the Euclidean time τ is confined to the interval $\tau \in [0, \beta]$, where $\beta = 1/T$. Periodic BC $\varphi(\tau + \beta, \mathbf{x}) = \varphi(\tau, \mathbf{x})$ for bosonic field are imposed in the imaginary time coordinate. The partition function \mathcal{Z} is given by

$$\mathcal{Z} = C \int D\varphi e^{-S_E[\varphi]}, \quad (5.1)$$

where $S_E[\varphi]$ is the Euclidean action

$$S_E[\varphi] = \frac{1}{2} \int_0^\beta d\tau \int d^D \mathbf{x} \varphi K_E \varphi, \quad (5.2)$$

with $K_E = -\square_E$ and $\square_E = \frac{\partial^2}{\partial \tau^2} + \Delta$ being Euclidean wave operator. Then the Helmholtz free energy can be expressed as

$$F = -\frac{1}{\beta} \log(\mathcal{Z}) = \frac{1}{2\beta} \text{Tr} \log(K_E). \quad (5.3)$$

The configuration is still a p -dimensional hypercubic cavity with the size $L_1 = L_2 = \dots = L_p = L$ and with the sizes of the left $(D - p)$ -dimension $L_{p+1}, L_{p+2}, \dots, L_D \gg L$ in $(D + 1)$ -dimensional spacetime. When the scalar field satisfies periodic, Dirichlet and Neumann BCs, the Helmholtz free energies have the following expressions, respectively

$$\begin{aligned} F^{(P)} = & \frac{1}{2\beta} \left(\prod_{j=p+1}^D \frac{L_j}{2\pi} \right) \sum_{\substack{n_0 \in \mathbb{Z} \\ \vec{n} \in \mathbb{Z}^p}} \int_{-\infty}^{\infty} d^{D-p} \mathbf{r} \\ & \times \log \left[\left(\frac{2\pi n_0}{\beta} \right)^2 + \left(\frac{2\pi n_1}{L} \right)^2 + \dots + \left(\frac{2\pi n_p}{L} \right)^2 + \mathbf{r}^2 \right], \end{aligned} \quad (5.4)$$

$$\begin{aligned} F^{(D)} = & \frac{1}{2\beta} \left(\prod_{j=p+1}^D \frac{L_j}{\pi} \right) \sum_{\substack{n_0 \in \mathbb{Z} \\ \vec{n} \in \mathbb{N}^p}} \int_{-\infty}^{\infty} d^{D-p} \mathbf{r} \\ & \times \log \left[\left(\frac{2\pi n_0}{\beta} \right)^2 + \left(\frac{\pi n_1}{L} \right)^2 + \dots + \left(\frac{\pi n_p}{L} \right)^2 + \mathbf{r}^2 \right], \end{aligned} \quad (5.5)$$

$$\begin{aligned} F^{(N)} = & \frac{1}{2\beta} \left(\prod_{j=p+1}^D \frac{L_j}{\pi} \right) \sum_{\substack{n_0 \in \mathbb{Z} \\ \vec{n} \in (\mathbb{N} \cup \{0\})^p}} \int_{-\infty}^{\infty} d^{D-p} \mathbf{r} \\ & \times \log \left[\left(\frac{2\pi n_0}{\beta} \right)^2 + \left(\frac{\pi n_1}{L} \right)^2 + \dots + \left(\frac{\pi n_p}{L} \right)^2 + \mathbf{r}^2 \right]. \end{aligned} \quad (5.6)$$

On the regularization of the divergency, Mellin transformation and zeta function technique still come in handy. The density of the free energy of the periodic case

$$\begin{aligned}
 f^{(P)} &\equiv \frac{F^{(P)}}{\prod_{j=p+1}^D L_j} \\
 &= -\frac{1}{2^{D-p}\pi^{\frac{D-p}{2}}\Gamma(\frac{D-p}{2})\beta} \\
 &\quad \times \sum_{\substack{n_0 \in \mathbb{Z} \\ \vec{n} \in \mathbb{Z}^p}} \lim_{s \rightarrow 0} \frac{\partial}{\partial s} \int_0^\infty \frac{r^{D-p-1}}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-[(\frac{2\pi n_0}{\beta})^2 + \frac{4\pi^2 \vec{n}^2}{L^2} + r^2]t} dt dr.
 \end{aligned} \tag{5.7}$$

With the Poisson summation eq.(2.12) employed on the n_0 summation, eq.(5.7) becomes

$$f^{(P)} = -\frac{1}{2^{D-p+1}\pi^{\frac{D-p}{2}}} \sum_{\substack{m_0 \in \mathbb{Z} \\ \vec{n} \in \mathbb{Z}^p}} \int_0^\infty t^{-1-\frac{D-p}{2}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{m_0^2 \beta^2}{4t}} e^{-\frac{4\pi^2 \vec{n}^2}{L^2} t} dt. \tag{5.8}$$

The terms $m_0 = 0$ can be taken out of the summation and $f^{(P)}$ is divided into two parts: the zero temperature part $\varepsilon_0^{(P)}$ and the temperature-dependent part $f_T^{(P)}$:

$$\varepsilon_0^{(P)} = -\frac{1}{2L^{D-p+1}} \pi^{\frac{D-p+1}{2}} \Gamma(-\frac{D-p+1}{2}) Z_p(p-D-1), \tag{5.9}$$

$$f_T^{(P)} = -\frac{2}{(\beta L)^{\frac{D-p+1}{2}}} \sum_{\substack{m_0 \in \mathbb{N} \\ \vec{n} \in \mathbb{Z}^p}} \left(\frac{\sqrt{\vec{n}^2}}{m_0}\right)^{\frac{D-p+1}{2}} K_{\frac{D-p+1}{2}}\left(\frac{2\pi m_0 \sqrt{\vec{n}^2} \beta}{L}\right). \tag{5.10}$$

The former can also be obtained by taking β in eq.(5.7) to infinity and turning the summation over n_0 into a integral. As done in previous sections, the regularized zero point energy density is

$$\varepsilon_0^{(P),\text{reg.}} = -\frac{\Gamma(\frac{D+1}{2}) Z_p(D+1)}{2\pi^{\frac{D+1}{2}} L^{D-p+1}}. \tag{5.11}$$

As for the temperature-dependent part (5.10), it is already finite for a given side length L . This is the very reason that the regularization of this part was neglected in some previous papers. It is not difficult to find that the free energy density and further the Casimir force density are divergent when L goes large enough, which contradicts to the fact that the Casimir force should tend to zero with the increase of the side lengths. So this part of the free energy density is yet to be regularized. Before that, the divided two parts of the free energy density for the other two BCs are given as follows,

$$\varepsilon_0^{(D/N),\text{reg.}} = -\frac{1}{2^{D+2} L^{D-p+1}} \sum_{q=0}^{p-1} (\mp 1)^q \frac{C_p^q \Gamma(\frac{D-q+1}{2})}{\pi^{\frac{D-q+1}{2}}} Z_{p-q}(D-q+1), \tag{5.12}$$

$$f_T^{(D/N)} = -\frac{2}{(2\beta L)^{\frac{D-p+1}{2}}} \sum_{\substack{m_0 \in \mathbb{N} \\ \vec{n} \in \mathbb{N}^p / \vec{n} \in (\mathbb{N} \cup \{\vec{0}\})^p}} \left(\frac{\sqrt{\vec{n}^2}}{m_0}\right)^{\frac{D-p+1}{2}} K_{\frac{D-p+1}{2}}\left(\frac{\pi m_0 \sqrt{\vec{n}^2} \beta}{L}\right) \tag{5.13}$$

5.2. The Regularization of the Temperature-Dependent Part

According to Sect. 2, it is safe to use “different” approach rather than the zeta function technique to regularize the temperature-dependent part of the free energy. The Abel-Plana formula eq.(2.6) is the choice here. For the cases here, since in eq.(2.6) the last term is convergent (and will be denoted as C in the following), the divergent integral is of concern. With the summation to be regularized denoted as $A \equiv \sum_{n=1}^{\infty} u(n)$, the regularized A will be

$$\begin{aligned} A^{\text{reg.}} &= \sum_{n=1}^{\infty} u(n) - \int_0^{\infty} u(x) dx \\ &= A - \int_0^{\infty} u(x) dx. \end{aligned} \quad (5.14)$$

5.2.1. Periodic BCs

Denoting

$$g(z) \equiv -2 \sum_{m_0 \in \mathbb{N}} \left(\frac{\sqrt{z}}{m_0 \beta L} \right)^{\frac{D-p+1}{2}} K_{\frac{D-p+1}{2}} \left(\frac{2\pi m_0 \sqrt{z} \beta}{L} \right), \quad (5.15)$$

then $f_T^{(P)} = \sum_{\vec{n} \in \mathbb{Z}^p} g(\vec{n}^2)$. According to Abel-Plana formula (2.6),

$$\begin{aligned} f_T^{(P)} &= \sum_{\vec{n} \in \mathbb{Z}^p} g(\vec{n}^2) = 2 \sum_{\substack{\vec{n} \in \mathbb{Z}^{p-1} \\ k \in \mathbb{N}}} g(\vec{n}^2 + k^2) + \sum_{\vec{n} \in \mathbb{Z}^{p-1}} g(\vec{n}^2) \\ &= 2 \sum_{\vec{n} \in \mathbb{Z}^{p-1}} \left[-\frac{1}{2} g(\vec{n}^2) + \int_0^{\infty} g(\vec{n}^2 + x^2) dx \right] + \sum_{\vec{n} \in \mathbb{Z}^{p-1}} g(\vec{n}^2) + C \\ &= 2 \sum_{\vec{n} \in \mathbb{Z}^{p-1}} \int_0^{\infty} g(\vec{n}^2 + x^2) dx + C \\ &= 2^p \int_0^{\infty} g(x_1^2 + \cdots + x_p^2) dx + C. \end{aligned} \quad (5.16)$$

That is

$$\begin{aligned} f_T^{(P)} &= -2^{p+1} \sum_{m_0 \in \mathbb{N}} \int_0^{\infty} \left(\frac{\sqrt{x^2}}{m_0 \beta L} \right)^{\frac{D-p+1}{2}} K_{\frac{D-p+1}{2}} \left(\frac{2m_0 \beta \pi \sqrt{x^2}}{L} \right) d^p \vec{x} + C \\ &= -\frac{L^p \Gamma(\frac{D+1}{2}) \zeta(D+1)}{\beta^{D+1} \pi^{\frac{D+1}{2}}} + C. \end{aligned} \quad (5.17)$$

Now, according to eq.(5.14) it is clearly seen that to get the regularized result, the term has to be subtracted from $f_T^{(P)}$ is

$$-\frac{L^p \Gamma(\frac{D+1}{2}) \zeta(D+1)}{\beta^{D+1} \pi^{\frac{D+1}{2}}}. \quad (5.18)$$

Then, the regularized temperature-dependent part of the free energy density is

$$f_T^{(P),\text{reg.}} = f_T^{(P)} + \frac{L^p \Gamma(\frac{D+1}{2}) \zeta(D+1)}{\beta^{D+1} \pi^{\frac{D+1}{2}}}. \quad (5.19)$$

5.2.2. Dirichlet and Neumann BCs

For Dirichlet and Neumann BCs, similarly, denoting

$$h(z) \equiv -2 \sum_{m_0 \in \mathbb{N}} \left(\frac{\sqrt{z}}{2\beta L m_0} \right)^{\frac{D-p+1}{2}} K_{\frac{D-p+1}{2}} \left(\frac{m_0 \pi \beta \sqrt{z}}{L} \right), \quad (5.20)$$

then

$$f_T^{(D)} = \sum_{\vec{n} \in \mathbb{N}^p} h(\vec{n}^2), \quad f_T^{(N)} = \sum_{\vec{n} \in (\mathbb{N} \cup \{\vec{0}\})^p} h(\vec{n}^2). \quad (5.21)$$

According to eq.(2.6)

$$\begin{aligned} f_T^{(D/N)} &= \sum_{\vec{n} \in \mathbb{N}^p / \vec{n} \in (\mathbb{N} \cup \{\vec{0}\})^p} h(\vec{n}^2) = \sum_{\vec{n} \in \mathbb{N}^{p-1} / \vec{n} \in (\mathbb{N} \cup \{\vec{0}\})^{p-1}} \sum_{k=1/\infty}^{\infty} h(\vec{n}^2 + k^2) \\ &= \sum_{\vec{n} \in \mathbb{N}^{p-1} / \vec{n} \in (\mathbb{N} \cup \{\vec{0}\})^{p-1}} \left[\mp \frac{1}{2} h(\vec{n}^2) + \int_0^\infty h(\vec{n}^2 + x^2) dx \right] + C. \end{aligned} \quad (5.22)$$

Following the procedure in Ref. 44 one has

$$\begin{aligned} f_T^{(D/N)} &= \sum_{q=0}^{p-1} C_p^q \left(\mp \frac{1}{2} \right)^q \int_0^\infty h(x_1^2 + \dots + x_{p-q}^2) d^{p-q} x + \left(\mp \frac{1}{2} \right)^p h(\vec{0}^2) + C \\ &= - \sum_{q=0}^{p-1} (\mp 1)^q \frac{C_p^q L^{p-q} \zeta(D-q+1)}{2^q \pi^{\frac{D-q+1}{2}} \beta^{D-q+1}} \Gamma\left(\frac{D-q+1}{2}\right) \\ &\quad - \frac{(\mp 1)^p \Gamma(\frac{D-p+1}{2})}{2^p \beta^{D-p+1} \pi^{\frac{D-p+1}{2}}} \zeta(D-p+1) + C. \end{aligned} \quad (5.23)$$

The related terms contributing to the divergency of the Casimir force, with which one is dealing in the first place, are

$$- \sum_{q=0}^{p-1} (\mp 1)^q \frac{C_p^q L^{p-q} \zeta(D-q+1)}{2^q \pi^{\frac{D-q+1}{2}} \beta^{D-q+1}} \Gamma\left(\frac{D-q+1}{2}\right), \quad (5.24)$$

which should be subtracted from the free energy. So, the regularized temperature-dependent parts of the free energy densities for these two BCs are

$$f_T^{(D/N),\text{reg.}} = f_T^{(D/N)} + \sum_{q=0}^{p-1} (\mp 1)^q \frac{C_p^q L^{p-q} \zeta(D-q+1)}{2^q \pi^{\frac{D-q+1}{2}} \beta^{D-q+1}} \Gamma\left(\frac{D-q+1}{2}\right). \quad (5.25)$$

In the case of periodic BCs, only one term needs to be subtracted, namely eq.(5.18), while in the cases of Dirichlet and Neumann BCs, there are p terms as shown in eq.(5.24). For $D = p = 3$, from eq.(5.18), one gets

$$-\frac{L^3\pi^2T^4}{90}, \quad (5.26)$$

and from eq.(5.24), one gets

$$-\frac{L^3\pi^2T^4}{90}, \quad \pm \frac{3\zeta(3)L^2T^3}{4\pi}, \quad -\frac{L\pi T^2}{8}, \quad (5.27)$$

where the sign “+” corresponds to Dirichlet BCs and the sign “−” to Neumann BCs. It is obvious that the term proportional to T^4 is the blackbody radiation energy restricted in the volume L^3 , regardless of the BCs. For Dirichlet BCs the three terms in (5.27) are the results obtained in the previous papers.^{150,156,157} Therefore, it can be said that eqs.(5.18) and (5.24) are the general results of the subtraction to get the physical Casimir free energy density for p -dimensional hypercube in $(D + 1)$ -dimensional spacetime.

The explicit expression of the terms to be subtracted here are indeed of order equal to or more than the square of the temperature as suggested in Refs. 158, 135. As emphasized before, when the side length tends to infinity the unregularized temperature-dependent part of the free energy density is divergent, and now one can see the terms to be subtracted are exactly proportional to the powers of the side length.

At this point, we write out the expressions of the physical free energy for all three kinds of BCs:

$$\begin{aligned} f^{(\text{P}),\text{Phys.}} &= \varepsilon_0^{(\text{P}),\text{reg.}} + f_T^{(\text{P}),\text{reg.}} \\ &= -\frac{\Gamma(\frac{D+1}{2})Z_p(D+1)}{2\pi^{\frac{D+1}{2}}L^{D-p+1}} - 2 \sum_{\substack{m_0 \in \mathbb{N} \\ \vec{n} \in \mathbb{Z}^p}} \left(\frac{\sqrt{\vec{n}^2}}{m_0\beta L} \right)^{\frac{D-p+1}{2}} K_{\frac{D-p+1}{2}} \left(\frac{2m_0\beta\pi\sqrt{\vec{n}^2}}{L} \right) \\ &\quad + \frac{L^p\zeta(D+1)\Gamma(\frac{D+1}{2})}{\pi^{\frac{D+1}{2}}\beta^{D+1}}, \end{aligned} \quad (5.28)$$

and

$$\begin{aligned}
f^{(\text{D/N}), \text{Phys.}} &= \varepsilon_0^{(\text{D/N}), \text{reg.}} + f_T^{(\text{D/N}), \text{reg.}} \\
&= -\frac{1}{2^{D+2} L^{D-p+1}} \sum_{q=0}^{p-1} \frac{C_p^q (\mp 1)^q \Gamma(\frac{D-q+1}{2})}{\pi^{\frac{D-q+1}{2}}} Z_{p-q}(D-q+1) \\
&\quad - 2 \sum_{m_0 \in \mathbb{N}} \sum_{\vec{n} \in \mathbb{N}^p / \vec{n} \in (\mathbb{N} \cup \{\vec{0}\})^p} \left(\frac{\sqrt{\vec{n}^2}}{2\beta L m_0} \right)^{\frac{D-p+1}{2}} K_{\frac{D-p+1}{2}} \left(\frac{m_0 \pi \beta \sqrt{\vec{n}^2}}{L} \right) \\
&\quad + \sum_{q=0}^{p-1} (\mp 1)^q \frac{C_p^q L^{p-q} \zeta(D-q+1)}{2^q \pi^{\frac{D-q+1}{2}} \beta^{D-q+1}} \Gamma\left(\frac{D-q+1}{2}\right).
\end{aligned} \tag{5.29}$$

5.3. Alternative Expressions of the Casimir Free Energy

The terms like $K_\nu(\alpha \frac{\beta}{L})$ in eqs.(5.10) and (5.25), which describe the cases of low temperature or small separations better since $K_\nu(z)$ converges fast for large z , come from the employment of the Poisson summation formula over the n_0 summation. One can also employ the Poisson summation formula over the \vec{n} summations, which will results in terms like $K_{\nu'}(\alpha' \frac{L}{\beta})$ that describe the cases of high temperature or large separations better. The two kinds of results are equivalent for the same case, and should be called the low temperature and high temperature expansions respectively for the only difference lies in the converging rapidness in different temperature regimes. Through the similar procedure, one can get the following high temperature expansions of the free energy densities:

$$\begin{aligned}
f'^{(P)} &= -\frac{\Gamma(\frac{D}{2})}{2\beta L^{D-p} \pi^{\frac{D}{2}}} Z_p(D) - \frac{2L^p}{\beta} \sum_{\substack{n_0 \in \mathbb{N} \\ \vec{m} \in \mathbb{Z}^p \setminus \{\vec{0}\}}} \left(\frac{n_0}{\beta L \sqrt{\vec{m}^2}} \right)^{\frac{D}{2}} K_{\frac{D}{2}} \left(\frac{2n_0 \pi L}{\beta} \sqrt{\vec{m}^2} \right) \\
&\quad - \frac{L^p \zeta(D+1) \Gamma(\frac{D+1}{2})}{\pi^{\frac{D+1}{2}} \beta^{D+1}},
\end{aligned} \tag{5.30}$$

$$\begin{aligned}
f'^{(D/N)} = & -\frac{(\mp 1)^p \Gamma(\frac{D-p+1}{2})}{2^p \beta^{D-p+1} \pi^{\frac{D-p+1}{2}}} \zeta(D-p+1) \\
& -\frac{1}{2^{D+1} \beta L^{D-p}} \sum_{q=0}^{p-1} \frac{(\mp 1)^q C_p^q \Gamma(\frac{D-q}{2})}{\pi^{\frac{D-q}{2}}} Z_{p-q}(D-q) \\
& -\frac{1}{2^{D-1} \beta} \sum_{q=0}^{p-1} C_p^q (\mp 1)^q L^{p-q} \sum_{\substack{n_0 \in \mathbb{N} \\ \vec{m} \in \mathbb{Z}^{p-q} \setminus \{\vec{0}\}}} \left(\frac{2n_0}{L\beta\sqrt{\vec{m}^2}} \right)^{\frac{D-q}{2}} K_{\frac{D-q}{2}} \left(\frac{4\pi n_0 L \sqrt{\vec{m}^2}}{\beta} \right) \\
& -\sum_{q=0}^{p-1} (\mp 1)^q \frac{C_p^q L^{p-q} \zeta(D-q+1)}{2^q \pi^{\frac{D-q+1}{2}} \beta^{D-q+1}} \Gamma\left(\frac{D-q+1}{2}\right).
\end{aligned} \tag{5.31}$$

Since the two expansions of the free energy densities in low and high temperature regimes are equivalent, finite physical results should be obtained from both expansions when $L \rightarrow \infty$. Now, in high temperature expansions (5.30) and (5.31), it is easy to see that the divergent terms as $L \rightarrow \infty$ are the last terms of each equation. So these terms, which coincide with eqs.(5.18) and (5.24), have to be removed. Then, the physical free energy densities in high temperature regime are expressed as

$$\begin{aligned}
f'^{(P),\text{Phys.}} = & -\frac{\Gamma(\frac{D}{2})}{2\beta L^{D-p} \pi^{\frac{D}{2}}} Z_p(D) \\
& -\frac{2L^p}{\beta^{\frac{D}{2}+1} L^{\frac{D}{2}}} \sum_{\substack{n_0 \in \mathbb{N} \\ \vec{m} \in \mathbb{Z}^p \setminus \{\vec{0}\}}} n_0^{\frac{D}{2}} (\vec{m}^2)^{-\frac{D}{4}} K_{\frac{D}{2}} \left(\frac{2n_0 \pi L}{\beta} \sqrt{\vec{m}^2} \right),
\end{aligned} \tag{5.32}$$

and

$$\begin{aligned}
f'^{(D/N),\text{Phys.}} = & -\frac{(\mp 1)^p \Gamma(\frac{D-p+1}{2})}{2^p \beta^{D-p+1} \pi^{\frac{D-p+1}{2}}} \zeta(D-p+1) \\
& -\frac{1}{2^{D+1} \beta L^{D-p}} \sum_{q=0}^{p-1} \frac{(\mp 1)^q C_p^q \Gamma(\frac{D-q}{2})}{\pi^{\frac{D-q}{2}}} Z_{p-q}(D-q) \\
& -\left[\frac{1}{2^{D-1} \beta} \sum_{q=0}^{p-1} C_p^q (\mp 1)^q L^{p-q} \right. \\
& \times \left. \sum_{\substack{n_0 \in \mathbb{N} \\ \vec{m} \in \mathbb{Z}^{p-q} \setminus \{\vec{0}\}}} \left(\frac{2n_0}{L\beta\sqrt{\vec{m}^2}} \right)^{\frac{D-q}{2}} K_{\frac{D-q}{2}} \left(\frac{4\pi n_0 L \sqrt{\vec{m}^2}}{\beta} \right) \right].
\end{aligned} \tag{5.33}$$

5.4. The Closed Case of $D = p$

The closed case of $D = p$ has been investigated by Ambjørn and Wolfram¹² and Lim and Teo³⁶ for periodic BCs. By making some modifications of the results of

$D > p$ case, we review both the low and high temperature expansions for all three BCs.

5.4.1. Low Temperature Expansion

From eqs.(5.28) and (5.29), when $\vec{n} \in \{\vec{0}\}$, the second terms becomes (for periodic and Neumann BCs but not for Dirichlet)

$$- \frac{\Gamma(\frac{D-p+1}{2})\zeta(D-p+1)}{\pi^{\frac{D-p+1}{2}}\beta^{D-p+1}}, \quad (5.34)$$

which is divergent for $D = p$. However, Ambjørn and Wolfram¹² have argued physically that this term is the free Bose gas result and should not appear in the result of physical free energy of $D = p$ case. Therefore, with this term excluded, the physical free energies for a closed $D = p$ cavity in low temperature expansion are

$$\begin{aligned} f_{D=p}^{(P),\text{Phys.}} = & - \frac{\Gamma(\frac{D+1}{2})Z_D(D+1)}{2\pi^{\frac{D+1}{2}}L} - 2 \sum_{\substack{m_0 \in \mathbb{N} \\ \vec{n} \in \mathbb{Z}^D \setminus \{\vec{0}\}}} \left(\frac{\sqrt{\vec{n}^2}}{m_0\beta L}\right)^{\frac{1}{2}} K_{\frac{1}{2}}\left(\frac{2m_0\beta\pi\sqrt{\vec{n}^2}}{L}\right) \\ & + \frac{L^p\zeta(D+1)\Gamma(\frac{D+1}{2})}{\pi^{\frac{D+1}{2}}\beta^{D+1}}, \end{aligned} \quad (5.35)$$

and

$$\begin{aligned} f_{D=p}^{(D/N),\text{Phys.}} = & - \frac{1}{2^{D+2}L} \sum_{q=0}^{D-1} \frac{C_D^q(\mp 1)^q \Gamma(\frac{D-q+1}{2})}{\pi^{\frac{D-q+1}{2}}} Z_{D-q}(D-q+1) \\ & - 2 \sum_{m_0 \in \mathbb{N}} \sum_{\vec{n} \in \mathbb{N}^D / \vec{n} \in (\mathbb{N} \cup \{\vec{0}\})^D} \left(\frac{\sqrt{\vec{n}^2}}{2\beta L m_0}\right)^{\frac{1}{2}} K_{\frac{1}{2}}\left(\frac{m_0\pi\beta\sqrt{\vec{n}^2}}{L}\right) \\ & + \sum_{q=0}^{D-1} (\mp 1)^q \frac{C_D^q L^{p-q} \zeta(D-q+1)}{2^q \pi^{\frac{D-q+1}{2}} \beta^{D-q+1}} \Gamma\left(\frac{D-q+1}{2}\right). \end{aligned} \quad (5.36)$$

5.4.2. High Temperature Expansion

Eq.(5.34) is still to be removed in the high temperature expansion to obtain the physical free energies for $D = p$ case. However, it is not that straightforward to get to the results as in low temperature regime. From eq.(5.32) one can see that the first term is also divergent for $D = p$. So, we have to deal with two divergent terms now.

Recall that the Epstein zeta function can be written as eq.(2.13). Replacing the argument and it can be rewritten as

$$Z_p(s) = \frac{2}{\Gamma(\frac{s}{2})} \sum_{j=0}^{p-1} \pi^{\frac{j}{2}} \Gamma\left(\frac{s-j}{2}\right) \zeta(s-j) + \frac{4\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} \sum_{j=1}^{p-1} \sum_{\substack{m \in \mathbb{N} \\ \vec{k} \in \mathbb{Z}^j \setminus \{\vec{0}\}}} \left(\frac{\sqrt{\vec{k}^2}}{m}\right)^{\frac{s-j}{2}} K_{\frac{s-j}{2}}(2\pi m \sqrt{\vec{k}^2}). \quad (5.37)$$

When $s = p$, the divergency lies only in the $j = p - 1$ term of the first part. So, the first term of eq.(5.32) can be expressed as

$$\begin{aligned} -\frac{\Gamma(\frac{D}{2})Z_p(D)}{2\beta L^{D-p}\pi^{\frac{D}{2}}} &= -\frac{\Gamma(\frac{D-p+1}{2})\zeta(D-p+1)}{\beta L^{D-p}\pi^{\frac{D-p+1}{2}}} - \frac{1}{\beta L^{D-p}} \sum_{j=0}^{p-2} \frac{\Gamma(\frac{D-j}{2})\zeta(D-j)}{\pi^{\frac{D-j}{2}}} \\ &\quad - \frac{2}{\beta L^{D-p}} \sum_{j=1}^{p-1} \sum_{\substack{m \in \mathbb{N} \\ \vec{k} \in \mathbb{Z}^j \setminus \{\vec{0}\}}} \left(\frac{\sqrt{k^2}}{m}\right)^{\frac{D-j}{2}} K_{\frac{D-j}{2}}(2\pi m \sqrt{k^2}), \end{aligned} \quad (5.38)$$

where the first term of the RHS is divergent and the rest are convergent when $D \rightarrow p$. Now, together with eq.(5.34), the divergency can be expressed as

$$-\lim_{D \rightarrow p} \frac{\Gamma(\frac{D-p+1}{2})\zeta(D-p+1)}{\pi^{\frac{D-p+1}{2}}\beta} \left[\frac{1}{L^{D-p}} - \frac{1}{\beta^{D-p}} \right] = -\frac{1}{\beta} \log\left(\frac{\beta}{L}\right). \quad (5.39)$$

Collecting all the pieces, the free energy for periodic BCs of $D = p$ case in high temperature regime is

$$\begin{aligned} f_{D=p}^{(P), \text{Phys.}} &= -\sum_{j=0}^{D-2} \frac{\Gamma(\frac{D-j}{2})\zeta(D-j)}{\beta \pi^{\frac{D-j}{2}}} - \frac{2L^{\frac{D}{2}}}{\beta^{\frac{D}{2}+1}} \sum_{\substack{n_0 \in \mathbb{N} \\ \vec{m} \in \mathbb{Z}^D \setminus \{\vec{0}\}}} \left(\frac{n_0}{\sqrt{\vec{m}^2}}\right)^{\frac{D}{2}} K_{\frac{D}{2}}\left(\frac{2n_0\pi L}{\beta} \sqrt{\vec{m}^2}\right) \\ &\quad - \frac{2}{\beta} \sum_{j=1}^{D-1} \sum_{\substack{m \in \mathbb{N} \\ \vec{k} \in \mathbb{Z}^j \setminus \{\vec{0}\}}} \left(\frac{\sqrt{k^2}}{m}\right)^{\frac{D-j}{2}} K_{\frac{D-j}{2}}(2\pi m \sqrt{k^2}) - \frac{1}{\beta} \ln \frac{\beta}{L}. \end{aligned} \quad (5.40)$$

For Dirichlet and Neumann BCs, as $D \rightarrow p$, the first two terms in eqs.(5.33) are both divergent. Through the procedure similar to eqs.(5.38) to (5.40), the physical free energy for $D = p$ case in Dirichlet and Neumann BCs is given as

$$\begin{aligned} f_{D=p}^{(D/N), \text{Phys.}} &= -\frac{1}{2^{D-1}\beta} \sum_{q=0}^{D-1} C_D^q (\mp 1)^q \sum_{\substack{n_0 \in \mathbb{N} \\ \vec{m} \in \mathbb{Z}^{D-q} \setminus \{\vec{0}\}}} \left(\frac{2n_0 L}{\beta \sqrt{\vec{m}^2}}\right)^{\frac{D-q}{2}} K_{\frac{D-q}{2}}\left(\frac{4\pi n_0 L \sqrt{\vec{m}^2}}{\beta}\right) \\ &\quad - \frac{1}{2^D \beta} \sum_{q=0}^{D-2} \sum_{j=0}^{D-q-2} \frac{(\mp 1)^q C_D^q}{\pi^{\frac{D-q-j}{2}}} \Gamma\left(\frac{D-q-j}{2}\right) \zeta(D-q-j) \\ &\quad - \frac{1}{2^{D-1}\beta} \sum_{q=0}^{D-2} \sum_{j=1}^{D-q-1} \sum_{\substack{m \in \mathbb{N} \\ \vec{k} \in \mathbb{Z}^j \setminus \{\vec{0}\}}} (\mp 1)^q C_D^q \left(\frac{\sqrt{k^2}}{m}\right)^{\frac{D-q-j}{2}} K_{\frac{D-q-j}{2}}(2\pi m \sqrt{k^2}) \\ &\quad + A, \end{aligned} \quad (5.41)$$

with

$$A = \begin{cases} -\frac{(-1)^D}{2^D \beta} \log\left(\frac{2L}{\beta}\right) & \text{for Dirichlet BCs;} \\ -\frac{1-2^D}{2^D \beta} \log\left(\frac{2L}{\beta}\right) & \text{for Neumann BCs.} \end{cases} \quad (5.42)$$

The finite temperature cases of all three BCs of $D \geq p$ has been reviewed till now, expressed in eqs. (5.28), (5.29), (5.32), (5.33), (5.35), (5.36), and (5.40) - (5.42), where it is easy to find that the absolute value of the physical free energy (density) in every case is an increasing function of temperature. More easy observations of these results are their asymptotic behaviors in high temperature limits, which are listed in Table 3. In the table, only the term of the highest power of $\frac{1}{\beta}$ is retained. The high temperature limits of the free energy (density) for Dirichlet BCs with even p and for Neumann and periodic BCs both with $D > p$ are negative whereas they are positive for Dirichlet BCs with odd p and for Neumann and periodic BCs both with $D = p$. Standard units are also restored in the table, and it can be seen that with the definition of effective temperature, $k_B T_{\text{eff}} \equiv \hbar c/L$ for periodic BCs, and $2k_B T_{\text{eff}} \equiv \hbar c/L$ for the other two BCs,⁷ the high temperature limits of the free energy (density) in all the cases are proportional to $k_B T$ and do not depend on the Planck constant, i.e., the classical limits are achieved.

Table 3. High temperature limits of the free energy (density)

		Natural Units	Standard Units
periodic	$D > p$	$-\frac{\Gamma(\frac{D}{2})Z_p(D)}{2\pi^{\frac{D}{2}}\beta L^{D-p}}$	$-k_B T \frac{\Gamma(\frac{D}{2})Z_p(D)}{2\pi^{\frac{D}{2}}L^{D-p}}$
	$D = p$	$-\frac{1}{\beta} \log(\frac{\beta}{L})$	$k_B T \log(\frac{T}{T_{\text{eff}}})$
Neumann	$D > p$	$-\frac{\Gamma(\frac{D-p+1}{2})\zeta(D-p+1)}{2^p \beta^{D-p+1} \pi^{\frac{D-p+1}{2}}}$	$-k_B T \frac{\Gamma(\frac{D-p+1}{2})\zeta(D-p+1)}{2^D L^{D-p} \pi^{\frac{D-p+1}{2}}} (\frac{T}{T_{\text{eff}}})^{D-p}$
	$D = p$	$-\frac{1-2^D}{2^D \beta} \log(\frac{2L}{\beta})$	$k_B T \frac{2^D-1}{2^D} \log(\frac{T}{T_{\text{eff}}})$
Dirichlet	$D > p$	$-\frac{(-1)^p \Gamma(\frac{D-p+1}{2})\zeta(D-p+1)}{2^p \beta^{D-p+1} \pi^{\frac{D-p+1}{2}}}$	$-k_B T \frac{(-1)^p \Gamma(\frac{D-p+1}{2})\zeta(D-p+1)}{2^D L^{D-p} \pi^{\frac{D-p+1}{2}}} (\frac{T}{T_{\text{eff}}})^{D-p}$
	$D = p$	$-\frac{(-1)^D}{2^D \beta} \log(\frac{2L}{\beta})$	$-k_B T (-\frac{1}{2})^D \log(\frac{T}{T_{\text{eff}}})$

We have collected some numerical analysis of the behaviors of the free energy from Ref. 44. In Fig. 5, it is shown for Dirichlet BCs the free energy density as a function of temperature at the side length $L = 10\text{eV}^{-1}$ with typical dimensions. And in Fig. 6, it is shown the free energy as a function of the side length for $D = p$ cases, where the logarithmic behaviors are illustrated, which although is not seen analytically in the expression in Ref. 135, is shown in its numerical illustration. More numerical analyses of the behaviors of the energy can be found in Ref. 44.

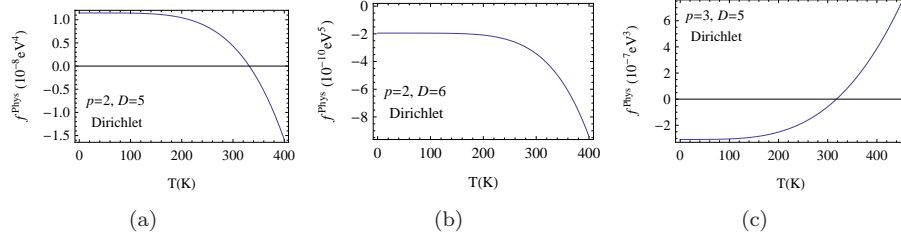


Fig. 5. The free energy density as a function of temperature at the side length $L = 10\text{eV}^{-1}$: (a) for Dirichlet BCs with $p = 2$, $D = 5$; (b) for Dirichlet BCs with $p = 2$, $D = 6$; (c) for Dirichlet BCs with $p = 3$, $D = 5$. Note that for $p = 2$, $D_{\text{crit}} = 6$ is the critical dimensionality of space for the zero point energy density given in eq.(5.12) to turn from positive to negative with D increasing.¹⁶

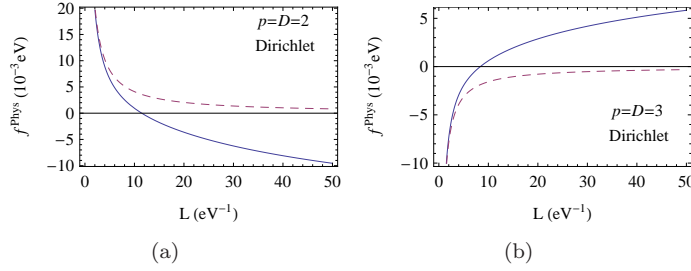


Fig. 6. The free energy as a function of the side length for $D = p$ cases. The solid lines are the free energies of $T = 300\text{K}$ and the dashed lines are the results of $T = 0\text{K}$. (a) is the cases of $D = p = \text{even}$ of Dirichlet BCs specifying $D = p = 2$. (b) is the cases $D = p = \text{odd}$ of Dirichlet or any $D = p$ cases of Neumann and periodic BCs, specifying $D = p = 3$ in Dirichlet.

6. Casimir Effect with Helix BCs

The Casimir effect arises not only in the presence of material boundaries, but also in spaces with nontrivial topology. For example, a flat manifold with the topology of S^1 causes the periodic condition $\phi(t, 0) = \phi(t, C)$.

Under the so-called helix boundary condition for a scalar field in 2+1 dimension is defined as

$$\phi(t, x + a, z) = \phi(t, x, z + h), \quad (6.1)$$

where h is regarded as the pitch of the helix, and this condition is called the helix BC. One can see that it would return to the cylindrical boundary conditions when h vanishes, but for $h \neq 0$, the whole system (the spring) does not have the cylindrical symmetry. In Sect. 6.1, we shall review this kind of boundary conditions from the lattices aspect in $(D + 1)$ dimensions.

The Casimir force on the x direction of the helix can be obtained by using the ζ function regularization:

$$\mathcal{F}^{(a)} = -\frac{\partial \mathcal{E}^{\text{reg.}}(a, h)}{\partial a} = -\frac{3\zeta(3)}{2\pi a^4} \left(1 + r^2\right)^{-5/2}, \quad (6.2)$$

which is always an attractive force and the magnitude of the force monotonously decreases with the increasing of the ratio r . Once r becomes large enough, the force can be neglected. While, the Casimir force on the z direction is

$$\mathcal{F}^{(h)} = -\frac{\partial \mathcal{E}^{\text{reg.}}(a, h)}{\partial h} = -\frac{3\zeta(3)}{2\pi a^4} \frac{r}{(1+r^2)^{5/2}}, \quad (6.3)$$

which has a maximum magnitude at $r \equiv h/a = 0.5$. When $r < 0.5$, the magnitude of the force increases with the increasing of r until $r = 0.5$. The Casimir force is almost linearly depending on r when $r \ll 1$, which is just like the force on a spring complying with the Hooke's law. However, in this case, the force originates from the quantum effect, namely, the Casimir effect. And then, it is called *quantum spring*. One shall see that the *quantum spring* can exist in any $(D+1)$ -dimensional spacetime.

There are also another interesting non-Euclidean topology BC inspired by Nanotubes, which could be given by introducing an arbitrary phase difference between $\phi(t, \mathbf{x} + \mathbf{a})$ and $\phi(t, \mathbf{x})$, namely,

$$\phi(t, \mathbf{x} + \mathbf{a}) = e^{i2\pi\theta} \phi(t, \mathbf{x}), \quad (6.4)$$

where the phase angle takes the value between $0 \leq \theta \leq 1$. Clearly, it will reduce to the (anti-) periodic BC when θ takes an (half-) integer value.

Generally, the phase could be any values besides -1 and 1 in the complex plane. For instance, when one considers the Casimir effect in nanotubes or nanoloops for a quantum field, $\theta = 0$ corresponds to metallic nanotubes, while $\theta = \pm 2\pi/3$ corresponds to semiconductor nanotubes. So, it is more reasonable and interesting to take the this kind of “quasi-periodic” BC (6.4) for the scalar field. In Ref. 43, the authors have studied the Casimire effect with the BC (6.4), and they found that an attractive or repulsive Casimir force could be arised depending on the values of the phase angle. Especially, the Casimir effect disappears when the phase angle takes a particular value. They have also investigated the high dimensional spacetime cases.

In $3+1$ dimensional spacetime that are most interested, the Casimir force could be obtained as

$$\mathcal{F}_3^{(a)} = \frac{4\pi^2}{3a^5} \left(-\frac{1}{30} + \theta^2 - 2\theta^3 + \theta^4 \right). \quad (6.5)$$

Clearly, the force is attractive or repulsive depending on the values of θ , and it could be even vanished when

$$\theta = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \frac{2\sqrt{30}}{15}}. \quad (6.6)$$

This phenomena is so interesting that it worth further studing, especially by combining the “quasi-periodic” BC (6.4) with the helix topological condition like Eq.(6.1):

$$\phi(t, x + a, z) = e^{i2\pi\theta} \phi(t, x, z + h). \quad (6.7)$$

In the rest of this section, we will review the Casimir effect with helix BCs in some cases.

6.1. *Scalar Casimir Effect*

6.1.1. *Topology of the Flat $(D+1)$ -Dimensional Spacetime*

Prior to the discussion of more complicated topology in the flat spacetime, it is beneficial to review the idea of lattices. A lattice Λ is defined as a set of points in a flat $(D+1)$ -dimensional spacetime \mathcal{M}^{D+1} , of the form

$$\Lambda = \left\{ \sum_{i=0}^D n_i \mathbf{e}_i \mid n_i \in \mathbb{Z} \right\}, \quad (6.8)$$

where $\{\mathbf{e}_i\}$ is a set of basis vectors of \mathcal{M}^{D+1} . In terms of the components v^i of vectors $\mathbf{V} \in \mathcal{M}^{D+1}$, the inner products is defined as

$$\mathbf{V} \cdot \mathbf{W} = \epsilon(a) v^i w^j \delta_{ij}, \quad (6.9)$$

with $\epsilon(a) = 1$ for $i = 0$, $\epsilon(a) = -1$ for otherwise. In the $x^1 - x^2$ plane, the sublattice $\Lambda'' \subset \Lambda' \subset \Lambda$ are

$$\Lambda' = \{ n_1 \mathbf{e}_1 + n_2 \mathbf{e}_2 \mid n_{1,2} \in \mathbb{Z} \}, \quad (6.10)$$

and

$$\Lambda'' = \{ n(\mathbf{e}_1 + \mathbf{e}_2) \mid n \in \mathbb{Z} \}. \quad (6.11)$$

The unit cylinder-cell is the set of points

$$U_c = \left\{ \mathbf{X} = \sum_{i=0}^D x^i \mathbf{e}_i \mid 0 \leq x^1 < a, -h \leq x^2 < 0, \right. \\ \left. -\infty < x^0 < \infty, -\frac{L}{2} \leq x^T \leq \frac{L}{2} \right\}, \quad (6.12)$$

where $T = 3, \dots, D$. When $L \rightarrow \infty$, it contains precisely one lattice point (i.e. $\mathbb{X} = 0$), and any vector \mathbb{V} has precisely one "image" in the unit cylinder-cell, obtained by adding a sublattice vector to it.

For the scalar Casimir effect, a topology of the flat $(D+1)$ -dimensional spacetime: $U_c \equiv U_c + \mathbf{u}$, $\mathbf{u} \in \Lambda''$ is considered. This topology causes the helix BCs for a massless or massive scalar field

$$\phi(t, x^1 + a, x^2, x^T) = \phi(t, x^1, x^2 + h, x^T), \quad (6.13)$$

where, if $a = 0$ or $h = 0$, it returns to the periodic BCs.

6.1.2. *Massless Scalar Field*

Under the BCs eq.(6.13) in $(D+1)$ -dimensional flat spacetime, the eigenfunctions of the massless scalar field satisfying the Klein-Gordon equation are

$$\phi_n(t, x^\alpha, x^T) = \mathcal{N} e^{-i\omega_n t + ik_x x + ik_z z + ik_T x^T}, \quad (6.14)$$

where $\alpha = 1, 2; T = 3, \dots, D$, \mathcal{N} is a normalization factor and $x^1 = x, x^2 = z$, and

$$\omega_n^2 = k_T^2 + k_x^2 + \left(-\frac{2\pi n}{h} + \frac{k_x}{h}a\right)^2 = k_T^2 + k_z^2 + \left(\frac{2\pi n}{a} + \frac{k_z}{a}h\right)^2. \quad (6.15)$$

Here, k_x and k_z satisfy

$$ak_x - hk_z = 2n\pi, (n = 0, \pm 1, \pm 2, \dots). \quad (6.16)$$

Thus the energy is given as

$$\mathcal{E}_D = \frac{1}{2a} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \sum_{n=-\infty}^{\infty} \sqrt{k_T^2 + k_z^2 + \left(\frac{2\pi n}{a} + \frac{k_z}{a}h\right)^2} \quad (6.17)$$

where it is assumed that $a \neq 0$ without losing generalities. Eq.(6.17) can be rewritten as

$$\begin{aligned} \mathcal{E}_D &= \frac{1}{2a\sqrt{\gamma}} \int \frac{d^{D-1}u}{(2\pi)^{D-1}} \sum_{n=-\infty}^{\infty} \sqrt{u^2 + \left(\frac{2\pi n}{a\gamma}\right)^2} \\ &= -\frac{\pi^{\frac{D}{2}}}{a^{D+1}\gamma^{\frac{D+1}{2}}} \Gamma\left(-\frac{D}{2}\right) \zeta(-D), \end{aligned} \quad (6.18)$$

with $\gamma \equiv 1 + \frac{h^2}{a^2}$. Eq.(6.18) can be regularized utilizing eq.(2.3). For $D = 2j + 1$,

$$\mathcal{E}_{2j+1}^{\text{reg.}} = -\frac{(2\pi)^{j+1}|B_{2j+2}|}{(2j+1)!!(2j+2)(a^2 + h^2)^{j+1}}, \quad (6.19)$$

where $j = 1, 2, \dots$ and the Bernoulli numbers are $B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}, B_{12} = -\frac{691}{2730}, B_{14} = \frac{7}{6}, B_{16} = -\frac{3617}{510}, \dots$. For $D = 2j$,

$$\mathcal{E}_{2j}^{\text{reg.}} = -\frac{(2j-1)!!\zeta(2j+1)}{(2\pi)^j(a^2 + h^2)^{j+\frac{1}{2}}} \quad (6.20)$$

The symmetry of $a \leftrightarrow h$ is obvious in both cases. It is worth noting that the Casimir energy has different expressions for the odd and even space dimensions.

The Casimir force on the x direction is

$$\mathcal{F}^{(a)} = -\frac{\partial \mathcal{E}_D^{\text{reg.}}}{\partial a} \quad (6.21)$$

In the case of odd-dimensional space, the Casimir force is calculated

$$\mathcal{F}_{2j+1}^{(a)} = -\frac{(2\pi)^{j+1}|B_{2j+2}|a}{(2j+1)!!(a^2 + h^2)^{j+2}}, \quad (6.22)$$

which has a maximum value of magnitude

$$\mathcal{F}_{2j+1}^{(a), \text{max}} = -\frac{(2\pi)^{j+1}|B_{2j+2}|}{(2j+1)!!h^{2j+3}} \sqrt{\frac{(2j+3)^{2j+3}}{(2j+4)^{2j+4}}} \quad (6.23)$$

at $a = \frac{h}{\sqrt{2j+3}}$. In the case of even-dimensional space, the Casimir force is

$$\mathcal{F}_{2j}^{(a)} = -\frac{2(j+\frac{1}{2})(2j-1)!!\zeta(2j+1)a}{(2\pi)^j(a^2 + h^2)^{j+\frac{3}{2}}}, \quad (6.24)$$

and the maximum value of force magnitude

$$\mathcal{F}_{2j}^{(a),\max} = -\frac{2(j+\frac{1}{2})(2j+1)!!\zeta(2j+1)}{(2\pi)^j h^{2j+2}} \sqrt{\frac{(2j+2)^{2j+2}}{(2j+3)^{2j+3}}} \quad (6.25)$$

is obtained at $a = \frac{h}{\sqrt{2j+2}}$. The force in both cases is attractive. The results for $\mathcal{F}^{(h)}$ are similar to those of $\mathcal{F}^{(a)}$ because of the symmetry between a and h .

In Table 4, the Casimir energy and forces in the two directions for $D = 2, 3, 4, 5$ are listed.

Table 4. The massless helix Casimir energy and forces of a scalar field for $D = 2, 3, 4, 5$.

D	$\mathcal{E}_D^{\text{reg.}}$	$\mathcal{F}_D^{(a)}$	$\mathcal{F}_D^{(h)}$
2	$-\frac{\zeta(3)}{2\pi} \frac{1}{(a^2+h^2)^{\frac{3}{2}}}$	$-\frac{3\zeta(3)}{2\pi} \frac{a}{(a^2+h^2)^{-\frac{5}{2}}}$	$-\frac{3\zeta(3)}{2\pi} \frac{h}{(a^2+h^2)^{-\frac{5}{2}}}$
3	$-\frac{\pi^2}{90} \frac{1}{(a^2+h^2)^2}$	$-\frac{2\pi^2}{45} \frac{a}{(a^2+h^2)^3}$	$-\frac{2\pi^2}{45} \frac{h}{(a^2+h^2)^3}$
4	$-\frac{3\zeta(5)}{4\pi^2} \frac{1}{(a^2+h^2)^{\frac{5}{2}}}$	$-\frac{15\zeta(5)}{4\pi^2} \frac{a}{(a^2+h^2)^{-\frac{7}{2}}}$	$-\frac{15\zeta(5)}{4\pi^2} \frac{h}{(a^2+h^2)^{-\frac{7}{2}}}$
5	$-\frac{2\pi^3}{945} \frac{1}{(a^2+h^2)^3}$	$-\frac{4\pi^3}{315} \frac{a}{(a^2+h^2)^4}$	$-\frac{4\pi^3}{315} \frac{h}{(a^2+h^2)^4}$

Fig. 7 is the illustration¹⁴⁵ of the behavior of the Casimir force on x direction in $D = 3$ dimension. The curves from the bottom to top correspond to $h = 0.9, 1.0, 1.1, 1.2$ respectively. It is clearly seen that the Casimir force decreases with h increasing and the maximum value of the force magnitude $\frac{2\pi^2}{45h^5} \sqrt{\frac{5^5}{6^6}}$ appears at $a = \frac{h}{\sqrt{5}}$.

Fig. 8 is the illustration¹⁴⁵ of the behavior of the Casimir force on x direction in different dimensions. The curves from the bottom to top correspond to $D = 2, 3, 4, 5$ respectively. It is set $h = 1.5$ in this figure. It is clearly seen that the Casimir force decreases with D increasing, and the value of a where the maximum value of the force is achieved also gets smaller with D increasing.

6.1.3. Massive scalar field

The massive case is slightly different in the eigenmodes with the mass μ

$$\begin{aligned} \omega_n^2 &= k_T^2 + k_x^2 + \left(-\frac{2\pi n}{h} + \frac{k_x}{h}a\right)^2 + \mu^2 \\ &= k_T^2 + k_z^2 + \left(\frac{2\pi n}{a} + \frac{k_z}{a}h\right)^2 + \mu^2. \end{aligned} \quad (6.26)$$

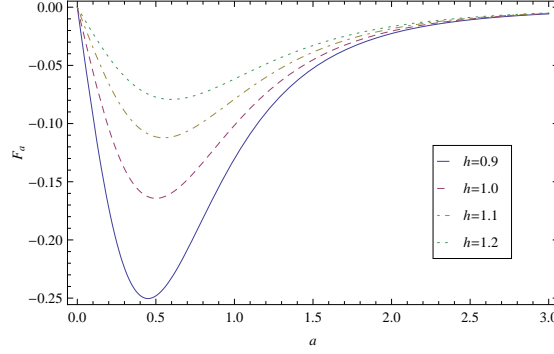


Fig. 7. The Casimir force on the x direction *vs.* a in $D = 3$ dimension for different h . The Casimir force decreases with h increasing and the maximum value of the force magnitude $\frac{2\pi^2}{45h^5} \sqrt{\frac{5^5}{6^6}}$ appears at $a = \frac{h}{\sqrt{5}}$.

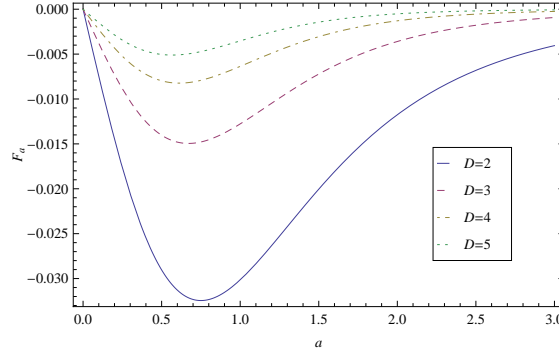


Fig. 8. The Casimir force on the x direction *vs.* a in different dimensions. It is set $h = 1.5$. It is clearly seen that the Casimir force decreases with D increasing, and the value of a where the maximum value of the force is achieved also gets smaller with D increasing.

where k_x and k_z satisfy eq.(6.16). The Casimir energy density of the massive scalar field in the $(D + 1)$ -dimensional spacetime is thus given by

$$\mathcal{E}_D^\mu = \frac{1}{2a} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \sum_{n=-\infty}^{\infty} \sqrt{k_T^2 + k_z^2 + \left(\frac{2\pi n}{a} + \frac{k_z}{a}h\right)^2 + \mu^2} \quad (6.27)$$

To regularize eq.(6.27), the functional relation

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} (bn^2 + \mu^2)^{-s} \\ &= \frac{\sqrt{\pi}}{\sqrt{b}} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \mu^{1-2s} + \frac{\pi^s}{\sqrt{b}} \frac{2}{\Gamma(s)} \sum_{n=-\infty}^{\infty} {}' \mu^{\frac{1}{2}-s} \left(\frac{n}{\sqrt{b}}\right)^{s-\frac{1}{2}} K_{\frac{1}{2}-s} \left(2\pi\mu \frac{n}{\sqrt{b}}\right) \end{aligned} \quad (6.28)$$

is used, where the prime means that the term $n = 0$ has to be excluded. After tedious deduction, one arrives

$$\mathcal{E}_D^{\mu, \text{reg.}} = -\frac{\mu^{D+1} \Gamma(-\frac{D+1}{2})}{2^{D+2} \pi^{\frac{D+1}{2}}} - 2 \left(\frac{\mu}{2\pi \sqrt{a^2 + h^2}} \right)^{\frac{D+1}{2}} \sum_{n=1}^{\infty} n^{-\frac{D+1}{2}} K_{\frac{D+1}{2}} \left(n\mu \sqrt{a^2 + h^2} \right). \quad (6.29)$$

Utilize the asymptotic behavior $K_\nu(z) \rightarrow \frac{2^{\nu-1} \Gamma(\nu)}{z^\nu}$ when $z \rightarrow 0$ for $\nu > 0$, it is not difficult to recover the result of the massless case when $\mu \rightarrow 0$.

Using $K'_\nu(z) = \frac{z}{K_\nu(z)} - K_{\nu+1}(z)$ where $K'_\nu(z) = dK_\nu(z)/dz$, one has the Casimir force

$$\mathcal{F}^{(a), \mu} = -\frac{2\mu a ((\mu a)^2 + (\mu h)^2)^{\frac{D+1}{4}}}{(2\pi)^{\frac{D+1}{2}} (a^2 + h^2)^{\frac{D+2}{2}}} \sum_{n=1}^{\infty} n^{-\frac{D-1}{2}} K_{\frac{D+3}{2}} \left(n\mu \sqrt{a^2 + h^2} \right). \quad (6.30)$$

Numerical analysis of the behavior of the Casimir force on x direction as a function of a for different h and D can be found in Ref. 145. The Casimir force is found still attractive and it has a maximum value similarly to massless case. For given values of D and μ , the behavior of the force for different h is similar to that in massless case. But for given values of h and μ , the behavior of the force for different D is opposite to that in massless case. The force increases with D increasing and the position of the maximum value move to larger a as D increasing. Fig. 9 shows the force as a function of a for $\mu = 1, h = 1$ and $D = 2, 3, 4, 5$ respectively and it is easy to find the difference between Fig. 9 and Fig. 8.

The rate of massive and massless cases is given as follows to study the precise way the Casimir force varies as the mass changes.

$$\frac{\mathcal{F}^{(a), \mu}}{\mathcal{F}^{(a), 0}} = \frac{((\mu a)^2 + (\mu h)^2)^{\frac{D+3}{4}}}{(D+1) 2^{\frac{D-1}{2}} \Gamma(\frac{D+1}{2}) \zeta(D+1)} \sum_{n=1}^{\infty} n^{-\frac{D-1}{2}} K_{\frac{D+3}{2}} \left(n\mu \sqrt{a^2 + h^2} \right). \quad (6.31)$$

In the case of odd-dimensional space, eq.(6.31) can be reduced to

$$\frac{\mathcal{F}^{(a), \mu}}{\mathcal{F}^{(a), 0}} = \frac{2(2j+1)!! ((\mu a)^2 + (\mu h)^2)^{\frac{j+2}{2}}}{(2\pi)^{2j+2} |B_{2j+2}|} \sum_{n=1}^{\infty} n^{-j} K_{j+2} \left(n\mu \sqrt{a^2 + h^2} \right), \quad (6.32)$$

where $j = 1, 2, \dots$. Obviously, the ratio tends to 1 when $\mu \rightarrow 0$ and it tends to zero when $\mu \rightarrow \infty$.

Fig. 10 is the illustration of the ratio of the Casimir force in massive case to that in massless case varying with the mass in $D = 3$ dimension. The curves correspond to $a = 1$ and $h = 0.1, 1, 2, 3$ respectively. Fig. 11 is the illustration of the ratio of the Casimir force in massive case to that in massless case varying with the mass for different dimensions. The curves correspond to $a = 1, h = 0.1$ and $D = 2, 3, 4, 5$ respectively. It is clearly seen from the two figures that the Casimir force decreases with μ increasing, and it approaches zero when μ tends to infinity. The plots also tell us the Casimir force for a massive field decreases with h increasing but it increases with D increasing. For the latter, the behavior of the Casimir force in massive case is different from that of massless case.

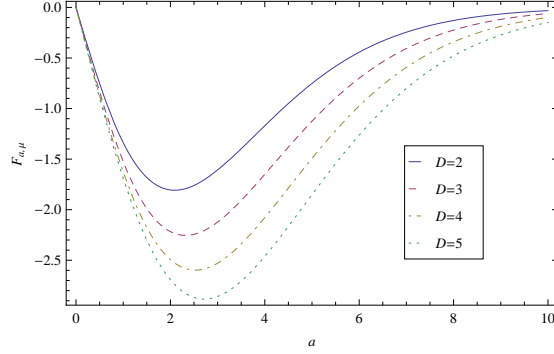


Fig. 9. The Casimir force on the x direction *vs.* a in different dimensions for a massive scalar field. It is taken $h = 1, \mu = 1$ and $D = 2, 3, 4, 5$ respectively. It is clearly seen that the Casimir force increases with D increasing, and the maximum value of the force moves to larger a as D increasing, which is a feature that opposite to massless case.

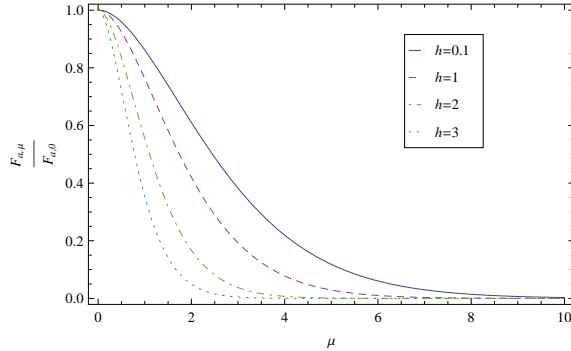


Fig. 10. the ratio of the Casimir force in massive case to that in massless case varying with the mass for different h in $D = 3$ dimension. The curves correspond to $a = 1$ and $h = 0.1, 1, 2, 3$ respectively.

6.2. Fermion Casimir Effect

6.2.1. The Vacuum Energy Density for a Fermion Field

For the fermion field, first a topological space X as follows

$$X = \bigcup_{\mathbf{u} \in \Lambda''} \{C_0 + \mathbf{u}\} \quad (6.33)$$

is considered in \mathcal{M}^{D+1} with the induced topology and define an equivalence relation \sim on X by

$$(x^1, x^2) \sim (x^1 - 2a, x^2 + 2h), \quad (6.34)$$

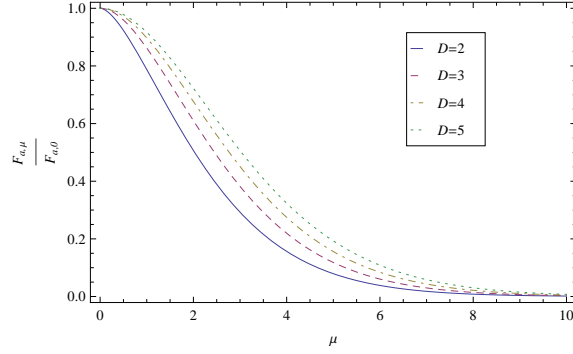


Fig. 11. the ratio of the Casimir force in massive case to that in massless case varying with the mass for different dimensions. We take $a = 1$ and $h = 0.1$.

then X/\sim with the quotient topology is homomorphic to helix topology. Here, Λ'' and unit cylinder-cell C_0 ^{143, 145} are

$$\Lambda'' = \{ n(\mathbf{e}_2 - \mathbf{e}_1) \mid n \in \mathbb{Z} \}. \quad (6.35)$$

and

$$C_0 = \left\{ \sum_{i=0}^D x^i \mathbf{e}_i \mid 0 \leq x^1 < a, -h \leq x^2 < 0, \right. \\ \left. -\infty < x^0 < \infty, -\frac{L}{2} \leq x^T \leq \frac{L}{2} \right\}, \quad (6.36)$$

where $T = 3, \dots, D$. Then the anti-helix conditions imposed on a field ψ ,

$$\psi(t, x^1 + a, x^2, x^T) = -\psi(t, x^1, x^2 + h, x^T) \quad (6.37)$$

is considered, where the field returns to the same value after traveling distances $2a$ at the x^1 -direction and $2h$ at the x^2 -direction. It is notable that a spinor wave function is anti-helix and takes its initial value after traveling distances $2a$ and $2h$ respectively. In other words, the anti-helix conditions are imposed on the field, which returns to the same field value $\psi(t, x^1 + 2a, x^2, x^T) = \psi(t, x^1, x^2 + 2h, x^T)$ only after two round trips. Therefore, the BC (6.37) can be induced by X/\sim with the quotient topology.

A spin-1/2 field $\psi(t, x^\alpha, x^T)$ defined in the $(D+1)$ -dimensional flat space-time satisfies the Dirac equation:

$$i\gamma^\mu \partial_\mu \psi - m_0 \psi = 0, \quad (6.38)$$

where $\alpha = 1, 2; T = 3, \dots, D$; $\mu = (t, \alpha, T)$ and m_0 is the mass of the Dirac field. γ^μ are $N \times N$ Dirac matrices with $N = 2^{[(D+1)/2]}$ where the square brackets mean the integer part of the enclosed expression. It is assumed in the following that these

matrices are given in the chiral representation:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k^+ & 0 \end{pmatrix}, k = 1, 2, \dots, D \quad (6.39)$$

with the relation $\sigma_\mu \sigma_\nu^+ + \sigma_\nu \sigma_\mu^+ = 2\delta_{\mu\nu}$. Under the BC eq.(6.37), the solutions of the field can be presented as

$$\psi^{(+)} = \mathcal{N}^{(+)} e^{-i\omega t} \begin{pmatrix} e^{i(k_x x + k_z z + k_T x^T)} \varphi_{(\alpha)} \\ -i\boldsymbol{\sigma}^+ \cdot \nabla e^{i(k_x x + k_z z + k_T x^T)} \varphi_{(\alpha)} / (\omega + m_0) \end{pmatrix}, \quad (6.40)$$

and

$$\psi^{(-)} = \mathcal{N}^{(-)} e^{i\omega t} \begin{pmatrix} i\boldsymbol{\sigma} \cdot \nabla e^{i(k_x x + k_z z + k_T x^T)} \chi_{(\alpha)} / (\omega + m_0) \\ e^{i(k_x x + k_z z + k_T x^T)} \chi_{(\alpha)} \end{pmatrix}, \quad (6.41)$$

where $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_D)$, $x^1 = x$, $x^2 = z$ and $\mathcal{N}^{(\pm)}$ is a normalization factor, and $\varphi_{(\alpha)}, \chi_{(\alpha)}$ are one-column constant matrices having $2^{[(D+1)/2]} - 1$ rows with the element $\delta_{\alpha\beta}$, $\alpha, \beta = 1, \dots, 2^{[(D+1)/2]} - 1$.

From eqs.(6.38)-(6.41), one has

$$\begin{aligned} \omega_n^2 &= k_T^2 + k_x^2 + \left(-\frac{2\pi(n + \frac{1}{2})}{h} + \frac{k_x}{h} a \right)^2 + m_0^2 \\ &= k_T^2 + k_z^2 + \left(\frac{2\pi(n + \frac{1}{2})}{a} + \frac{k_z}{a} h \right)^2 + m_0^2, \end{aligned} \quad (6.42)$$

with k_x and k_z satisfying

$$ak_x - hk_z = 2 \left(n + \frac{1}{2} \right) \pi, (n = 0, \pm 1, \pm 2, \dots). \quad (6.43)$$

The energy density of the field in $(D+1)$ -dimensional space-time is thus given by

$$\mathcal{E}_D = -\frac{N}{2a} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \sum_{n=-\infty}^{\infty} \sqrt{k_T^2 + k_z^2 + \left(\frac{2\pi(n + \frac{1}{2})}{a} + \frac{k_z}{a} h \right)^2 + m_0^2}, \quad (6.44)$$

where it is also assumed $a \neq 0$ without losing generalities.

Eq.(6.44) can be rewritten as

$$\begin{aligned} \mathcal{E}_D &= -\frac{N}{2a\sqrt{\gamma}} \int \frac{d^{D-1}u}{(2\pi)^{D-1}} \sum_{n=-\infty}^{\infty} \sqrt{u^2 + \left(\frac{2\pi(n + \frac{1}{2})}{a\sqrt{\gamma}} \right)^2 + m_0^2} \\ &= \frac{2^{[(D+1)/2]} - (D+1) \Gamma(-\frac{D}{2})}{\pi^{\frac{D}{2}} a \sqrt{\gamma}} \sum_{n=-\infty}^{\infty} \left[\left(\frac{2\pi(n + \frac{1}{2})}{a\sqrt{\gamma}} \right)^2 + m_0^2 \right]^{\frac{D}{2}}, \end{aligned} \quad (6.45)$$

with $\gamma \equiv 1 + \frac{h^2}{a^2}$.

It is seen from eq.(6.45) that the expression for the vacuum energy in the case of helix BCs can be obtained from the corresponding expression in the case of standard BC $\psi(t, x^1 + a, x^2, x^T) = -\psi(t, x^1, x^2, x^T)$ by making the change $a \rightarrow$

$a\sqrt{\gamma} = \sqrt{a^2 + h^2}$. The topological fermionic Casimir effect in toroidally compactified space-times has been recently investigated in Ref. 72 for non-helix BCs including general phases. In the limiting case $h = 0$, our result of eq.(6.45) is a special case of general formulas from Ref. 72.

6.2.2. The Case of Massless Field

For a massless Dirac field, that is, in the case of $m_0 = 0$, the energy density in eq.(6.45) is reduced to

$$\mathcal{E}_D^0 = \frac{2^{[(D+1)/2]} \pi^{\frac{D}{2}}}{a^{D+1} \gamma^{\frac{D+1}{2}}} \Gamma\left(-\frac{D}{2}\right) \zeta\left(-D, \frac{1}{2}\right). \quad (6.46)$$

where $\zeta(-D, \frac{1}{2})$ is the Hurwitz-Riemann ζ function. Using the relation

$$\zeta\left(s, \frac{1}{2}\right) = (2^s - 1) \zeta(s), \quad (6.47)$$

and the reflection relation eq.(2.3), the energy density can be regularized to be

$$\mathcal{E}_D^{0,\text{reg.}} = 2^{[(D+1)/2]} (2^{-D} - 1) \frac{\Gamma\left(\frac{D+1}{2}\right) \zeta(D+1)}{\pi^{\frac{D+1}{2}} (a^2 + h^2)^{\frac{D+1}{2}}}. \quad (6.48)$$

The Casimir force on the x direction is

$$\begin{aligned} \mathcal{F}_D^{0,(a)} &= - \frac{\partial \mathcal{E}_D^{0,\text{reg.}}}{\partial a} \\ &= 2^{[(D+1)/2]} (2^{-D} - 1) \frac{(D+1) \Gamma\left(\frac{D+1}{2}\right) \zeta(D+1)}{\pi^{\frac{D+1}{2}}} \frac{a}{(a^2 + h^2)^{\frac{D+3}{2}}}. \end{aligned} \quad (6.49)$$

It is obvious that the energy density is negative and the force is attractive. Furthermore, the force has a maximum value

$$\mathcal{F}_D^{0,(a),\text{max}} = 2^{[(D+1)/2]} (2^{-D} - 1) \frac{(D+1) \Gamma\left(\frac{D+1}{2}\right) \zeta(D+1)}{\pi^{\frac{D+1}{2}} h^{D+2}} \sqrt{\frac{(D+2)^{D+2}}{(D+3)^{D+3}}} \quad (6.50)$$

at $a = \frac{h}{\sqrt{D+2}}$. The results for $\mathcal{F}_D^{0,(h)}$ are similar to those of $\mathcal{F}_D^{0,(a)}$ because of the symmetry between a and h .

6.2.3. The Case of Massive Field

For a massive Dirac field, to regularize the series in eq.(6.45) the Chowla-Selberg formula is used directly¹⁵⁹

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \left[\frac{1}{2} a(n+c)^2 + b \right]^{-s} &= \frac{(2\pi)^{\frac{1}{2}} b^{\frac{1}{2}-s}}{\sqrt{a}} \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} + \frac{2^{\frac{s}{2} + \frac{1}{4} + 2} \pi^s b^{-\frac{s}{2} + \frac{1}{4}}}{\sqrt{a} \Gamma(s)} \\ &\quad \times \sum_{n=1}^{\infty} \cos(2\pi n c) \left(\frac{n^2}{a} \right)^{\frac{s}{2} - \frac{1}{4}} K_{\frac{1}{2}-s} \left(2\pi n \sqrt{\frac{2b}{a}} \right). \end{aligned} \quad (6.51)$$

Note that in the renormalization procedure, the vacuum energy in a flat space-time with trivial topology should be renormalized to zero, that is, in the expression for the renormalized vacuum energy the term corresponding to the first term in the right hand side of eq.(6.51) should be omitted. Thus, the Casimir energy has the expression as follows

$$\begin{aligned} \mathcal{E}_D^{m_0, \text{reg.}} = & 2^{[(D+3)/2]} \left(\frac{m_0}{2\pi\sqrt{a^2 + h^2}} \right)^{\frac{D+1}{2}} \\ & \times \sum_{n=1}^{\infty} \cos(\pi n) n^{-\frac{D+1}{2}} K_{\frac{D+1}{2}} \left(nm_0\sqrt{a^2 + h^2} \right). \end{aligned} \quad (6.52)$$

One can also find that the energy recovers the massless result by use of the asymptotic behavior of $K_\nu(z)$.

And similarly the Casimir force

$$\begin{aligned} \mathcal{F}^{(a), m_0} = & \frac{2^{[(D+3)/2]} m_0 a \left((m_0 a)^2 + (m_0 h)^2 \right)^{\frac{D+1}{4}}}{(2\pi)^{\frac{D+1}{2}} (a^2 + h^2)^{\frac{D+2}{2}}} \\ & \times \sum_{n=1}^{\infty} \cos(\pi n) n^{-\frac{D+1}{2}} K_{\frac{D+3}{2}} \left(nm_0\sqrt{a^2 + h^2} \right). \end{aligned} \quad (6.53)$$

The influence of mass on the Casimir force is similar to that of the case of scalar field. Numerical analysis can be found in Ref. 144.

We have reviewed in this section the scalar and fermionic quantum spring in $(D+1)$ -dimensional spacetime using the zeta function techniques. The Casimir force of both the scalar and spinor field is attractive, and has a maximum of magnitude. The influence of mass of the field on the Casimir effect has also been reviewed for both fields, and the precise way of the Casimir force changing with the mass has given.

7. Summary and Outlook

The quantum property of the reality, may be one of the most mystic, revolutionary but also fascinating concepts that physics theory has ever been brought to us. The Casimir effect provides a possibility to have a direct and perhaps macroscopic access to the insight of this reality, which makes the topic of this effect still full of vigor and vitality after its discovery more than 60 years ago. Study of the Casimir effect in rectangular boxes, one of the typical configuration of this topic, captures a lot of features of the effect.

In this article, we have reviewed several researches related to the Casimir effect in rectangular boxes. The frequently used regularization methods, the zeta function and Abel-Plana formula techniques, are proven identifiable, which gives our freedom to choose any regularization method at convenient. The equivalence of these two approaches may be extended to other regularization methods and other configurations. With the powerful zeta function technique, we have summarized the

attractive and repulsive nature of the Casimir effect in rectangular boxes with various settings, and in addition, reviewed the rectangular Casimir piston. Unlike the boxes, Casimir forces on the piston are always attractive no matter how the ratios of edges change. These researches have probed into the very nature of the quantum field on vacuum state. And furthermore, the study of the Casimir effect of quantum field on non-vacuum states, the equilibrium state characterized by a finite temperature has also been presented. The thermal Casimir effect, compared to the zero point energy, is a more practical and feasible subject, since an ensemble and distribution of excited states is a typical situation and almost all experiments are done under some temperature. The configuration of hypercube is only a simplified example to illustrate that both the zero temperature and temperature-dependent parts of the free energy need to be regularized. Temperature corrections on the effect for the configuration suggested in Sect. 3 and 4 are worth looking into. Finally, we have reviewed the Casimir effect arising from the non-trivial topology of the space, the quantum spring for both scalar and fermion fields, in which it is found that the forces are always attractive and have a maximum of magnitude. In practice, the study quantum spring may be applied to microelectromechanical system (MEMS).

Fragmental may be these researches, we have seen different aspects of the nature of the Casimir effect. These deepen understandings of the quantum nature, pieces of which may be united in future, have found their applications in various areas of research, including both fundamental physics and applied science. Extensive and detailed study of the subject suggests that its infancy is over and it is progressing towards its mature. We hope our review will serve as a collection of a part of the resource for its future development.

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